17.06.2015

Scritto Fisica Matematica 1

Esercizio $\emptyset$

$$\underline{F} = \begin{pmatrix} 2x \\ 2y \\ z^2 \end{pmatrix} \qquad \frac{\partial F_x}{\partial z} = x \qquad \frac{\partial F_z}{\partial x} = \emptyset \qquad \Rightarrow$$

La forza NON è conservativa.

Esercizio 1

(i) $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{cases} y = -x^2 \\ y = 4x^2 - x^4 \end{cases}$

$$x^4 - 5x^2 = 0 \qquad x^2(x^2-5) = 0$$

$$P_0 = (0,0) \qquad P_1 = (\sqrt{5}, -5) \qquad P_2 = (-\sqrt{5}, -5)$$

Calcolo la matrice Jacobiana

$$J(x,y) = \begin{pmatrix} 2x & 1 \\ 4x^3 - 8x & 1 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \lambda_1 = 0 \qquad \lambda_2 = 1$$

Equilibrio degenere (instabile)

$$J(\sqrt{5}, -5) = \begin{pmatrix} 2\sqrt{5} & 1 \\ 12\sqrt{5} & 1 \end{pmatrix}$$

$$(2\sqrt{5} - \lambda)(1-\lambda) - 12\sqrt{5} = 0$$

$$\lambda^2 - (1+2\sqrt{5})\lambda - 10\sqrt{5} = 0$$

$$\Delta = (1+2\sqrt{5})^2 + 40\sqrt{5} = 21 + 44\sqrt{5}$$

$$\lambda_{1,2} = \frac{(1+2\sqrt{5}) \pm \sqrt{(1+2\sqrt{5})^2 + 40\sqrt{5}}}{2} \qquad \lambda_1 > 0 \qquad \lambda_2 < 0$$

SELLA (instabile)

$$J(-\sqrt{5}, -5) = \begin{pmatrix} -2\sqrt{5} & 1 \\ -12\sqrt{5} & 1 \end{pmatrix}$$

$$(-2\sqrt{5} - \lambda)(1-\lambda) + 12\sqrt{5} = 0$$

$$\lambda^2 - (1-2\sqrt{5})\lambda + 10\sqrt{5} = 0$$

$$\Delta = (1-2\sqrt{5})^2 - 40\sqrt{5} = 21 - 44\sqrt{5} < 0$$

$$\lambda_{1,2} = \frac{1 - 2\sqrt{5} \pm i\sqrt{44\sqrt{5} - 21}}{2}$$

FUOCO STABILE (instabile)

(ii) Autovettori per $P_1 = (\sqrt{5}, -5)$

$$\begin{pmatrix} 2\sqrt{5} - \lambda_1 & 1 \\ 12\sqrt{5} & 1 - \lambda_1 \end{pmatrix} \begin{pmatrix} u_x^{(1)} \\ u_y^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u^{(1)} = \begin{pmatrix} 1 \\ \lambda_1 - 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{1 - 2\sqrt{5} + \sqrt{(1+2\sqrt{5})^2 + 40\sqrt{5}}}{2} \end{pmatrix}$$

Similmente

$$u^{(2)} = \begin{pmatrix} 1 \\ \lambda_2 - 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{1 - 2\sqrt{5} - \sqrt{(1+2\sqrt{5})^2 + 40\sqrt{5}}}{2} \end{pmatrix}$$

SOLUZIONE LINEARIZZATA

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \alpha\, e^{\lambda_1 t} + \beta\, e^{\lambda_2 t} \\ \alpha (\lambda_1 - 2\sqrt{5})\, e^{\lambda_1 t} + \beta (\lambda_2 - 2\sqrt{5})\, e^{\lambda_2 t} \end{pmatrix}$$

Autovettori per $P_2 = (-\sqrt{5}, -5)$

$$\begin{pmatrix} -2\sqrt{5} - \lambda_1 & 1 \\ -12\sqrt{5} & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} v_x^{(1)} \\ v_y^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v^{(1)} = \begin{pmatrix} 1 \\ \lambda_1 + 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+2\sqrt{5}}{2} \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\sqrt{44\sqrt{5}-21}}{2} \end{pmatrix}$$

Similmente

$$v^{(2)} = \begin{pmatrix} 1 \\ \frac{1+2\sqrt{5}}{2} \end{pmatrix} - i \begin{pmatrix} 0 \\ \frac{\sqrt{44\sqrt{5}-21}}{2} \end{pmatrix}$$

SOLUZIONE LINEARIZZATA

$\lambda_1 = \lambda_2^*$ $\qquad \lambda_1 = \frac{1-2\sqrt{5}}{2} + i\,\frac{\sqrt{44\sqrt{5}-21}}{2} = \mu + i\omega$

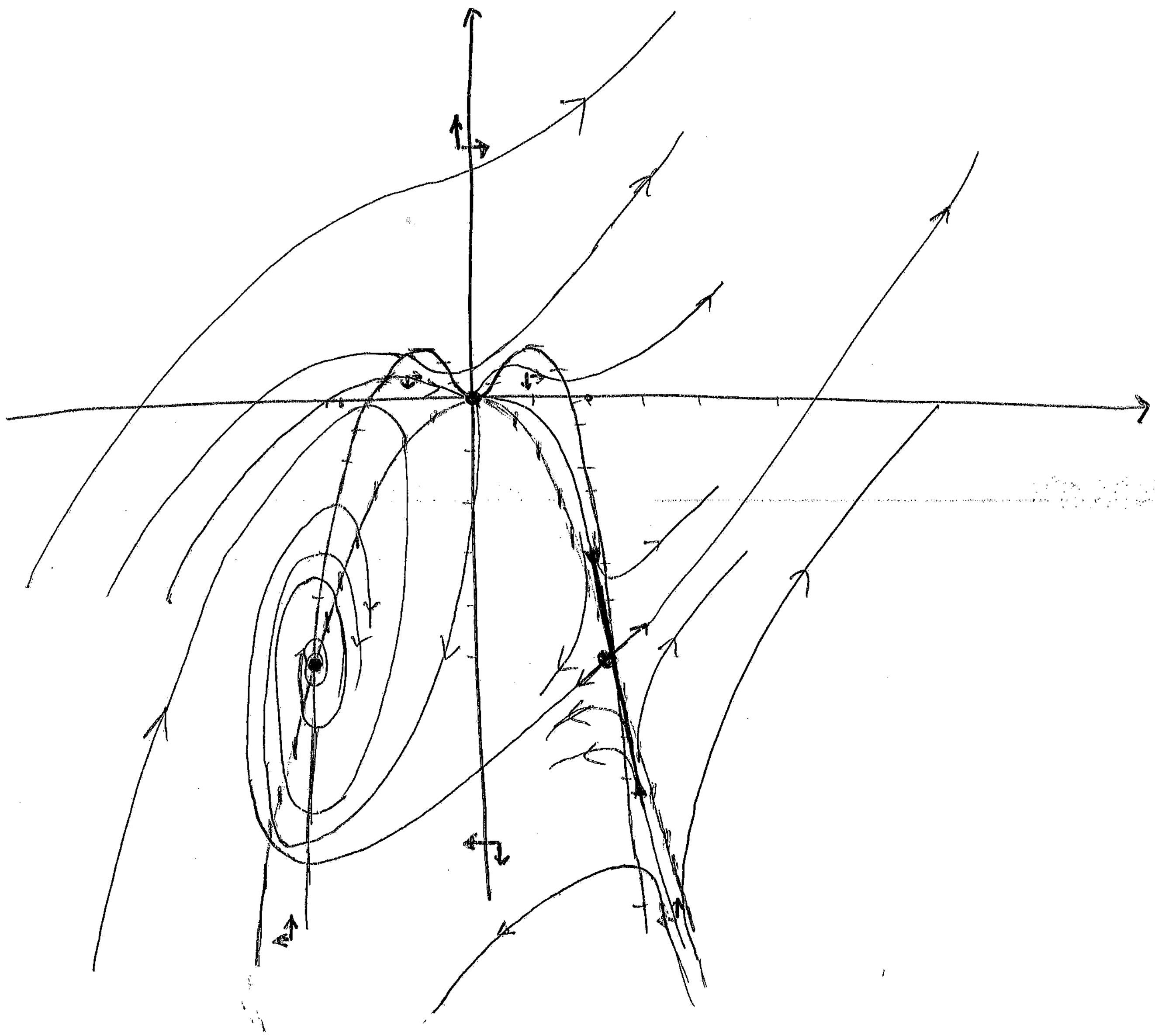
$v^{(1)} = v^{(2)*}$ $\qquad v^{(1)} = \begin{pmatrix} 1 \\ \frac{1+2\sqrt{5}}{2} \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\sqrt{44\sqrt{5}-21}}{2} \end{pmatrix} = \underline{u} + i\,\underline{v}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \rho e^{\mu t} \cos(\omega t + \varphi) \\ \rho e^{\mu t}\left(\frac{1+2\sqrt{5}}{2}\right)\cos(\omega t+\varphi) - \rho e^{\mu t}\left(\frac{\sqrt{44\sqrt{5}-21}}{2}\right)\sin(\omega t+\varphi) \end{pmatrix}$$

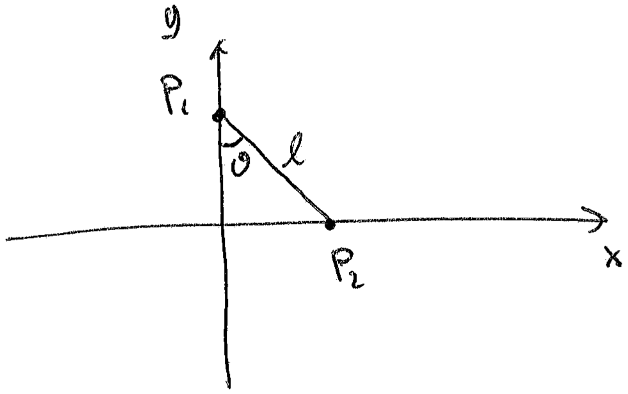
(iii) Per il ritratto di fase è conveniente analizzare le zeroclines

$\dot{x} = 0 \qquad y = -x^2 \qquad$ (tangente verticale)

$\dot{y} = 0 \qquad y = -x^4 + 4x^2 \qquad$ (tangente orizzontale)



Esercizio 2



$$P_1 : (0, \ell\cos\vartheta) \qquad \dot{P}_2 : (0, -\ell\sin\vartheta\,\dot{\vartheta})$$

$$P_2 : (\ell\sin\vartheta, 0) \qquad \dot{P}_2 : (\ell\cos\vartheta\,\dot{\vartheta}, 0)$$

$$T = \frac{1}{2} m \left(\ell^2 \dot{\vartheta}^2\right) \qquad V = mg\ell\cos\vartheta$$

$$L = \frac{1}{2} m \ell^2 \dot{\vartheta}^2 - mg\ell\cos\vartheta$$

$$\frac{\partial L}{\partial \dot{\vartheta}} = m\ell^2\dot{\vartheta} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} = m\ell^2\ddot{\vartheta} \qquad \frac{\partial L}{\partial \vartheta} = mg\ell\sin\vartheta$$

$$m\ell^2\ddot{\vartheta} = mg\ell\sin\vartheta$$

$$\boxed{\ddot{\vartheta} = \frac{g}{\ell}\sin\vartheta}$$

È equivalente al pendolo!

Equilibrio

$V'(\vartheta) = 0 \qquad \vartheta_1 = 0 \quad , \quad \vartheta_2 = \pi$

$P_1 = (0,0) \quad , \quad P_2 = (\pi, 0)$

$\vartheta_1 = 0$ è massimo del potenziale $\Rightarrow$ P_1 instabile

$\vartheta_2 = \pi$ è minimo del potenziale $\Rightarrow$ P_2 stabile

L'equazione linearizzata è

$$\ddot{\vartheta} = -\frac{g}{l}\vartheta \Rightarrow \omega = \sqrt{\frac{g}{l}}$$ FREQUENZA PICCOLE OSCILLAZIONI

Alternativamente

$m l^2 = A$ (termine cinetico)

$V''(\vartheta) = -mgl \cos\vartheta \qquad V''(\pi) = mgl = B$

$$A - \lambda B = 0 \qquad \lambda = \frac{B}{A} = \frac{g}{l} \qquad \boxed{\lambda = \omega^2}$$

Esercizio 3

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \\ z = \rho^4 + \rho^2 - 2 \end{cases} \qquad \begin{cases} \dot{x} = \dot\rho \cos\vartheta - \rho\dot\vartheta \sin\vartheta \\ \dot{y} = \dot\rho \sin\vartheta + \rho\dot\vartheta \cos\vartheta \\ \dot{z} = 4\rho^3\dot\rho + 2\rho\dot\rho = 2\rho(2\rho^2+1)\dot\rho \end{cases}$$

$$T = \frac{1}{2} m \left(\dot\rho^2 + \rho^2\dot\vartheta^2 + 4\rho^2(2\rho^2+1)^2\dot\rho^2 \right)$$

$$V = mg\left(\rho^4+\rho^2-2\right) + \frac{K}{2}\left(\rho^2 + \left(\rho^4+\rho^2-2\right)^2\right)$$

$$= mg\rho^4 + mg\rho^2 + \frac{K}{2}\left(\rho^2+\rho^8+\rho^4+2\rho^6-4\rho^4-4\rho^2\right) + (\text{costante})$$

$$= \frac{K}{2}\rho^8 + K\rho^6 + \left(mg - \frac{3}{2}K\right)\rho^4 + \left(mg-\frac{3}{2}K\right)\rho^2$$

$$L = \frac{1}{2} m \left(\dot\rho^2 + \rho^2\dot\vartheta^2 + 4\rho^2(2\rho^2+1)^2\dot\rho^2 \right) - \frac{K}{2}\rho^8 - K\rho^6$$
$$- \left(mg-\frac{3}{2}K\right)\rho^4 - \left(mg-\frac{3}{2}K\right)\rho^2$$

Eq. Lagrange (... esercizio)

θ è una variabile ciclica, quindi

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m\rho^2\dot\theta \qquad \text{(costante del moto)}$$

La seconda costante del moto è data dall'energia

$$E = T + V \qquad \text{(costante del moto)}$$

Sostituendo $\dot\theta = \dfrac{p_\theta}{m\rho^2}$ nell'energia, ottengo il potenziale efficace

$$V_{eff} = \frac{p_\theta^2}{2m\rho^2} + \frac{K}{2}\rho^8 + K\rho^6 + \left(mg - \frac{3}{2}K\right)\left(\rho^4 + \rho^2\right)$$

Studiamo quindi il grafico del potenziale efficace al variare dei parametri.

$\boxed{p_\theta = 0}$

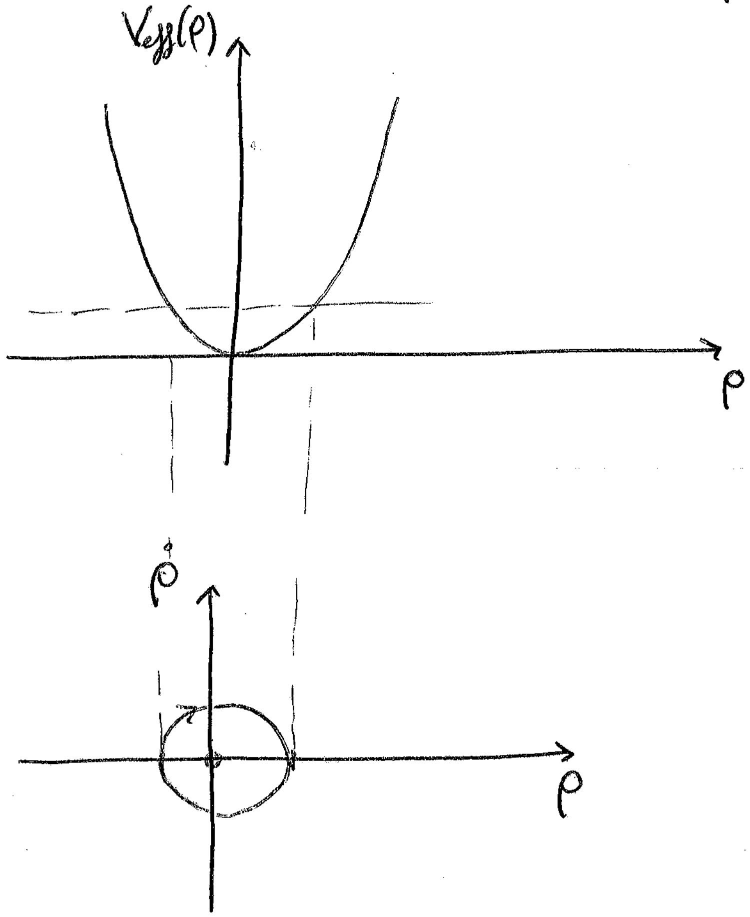
⚠ La dinamica si svolge su un meridiano, $\rho \in \mathbb{R}$ (non è più un raggio!).

$$V_{eff}(\rho) = \frac{K}{2}\rho^8 + K\rho^6 + \left(mg - \frac{3}{2}K\right)\left(\rho^4 + \rho^2\right)$$

Introduciamo $\boxed{\alpha = mg - \frac{3}{2}K}$

Se $\alpha \geq 0 \Rightarrow$ ho un solo equilibrio $(\rho = 0)$

$P_0 = (0, 0)$



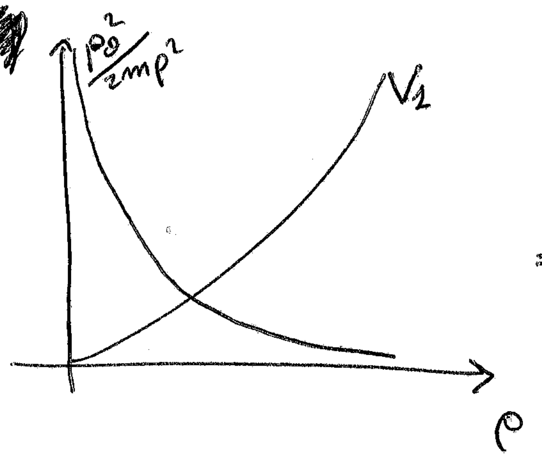
Discussione dettagliata per esercizio.

$\boxed{p_\theta \neq 0}$ ⚠ Ora $\rho > 0$, è un raggio!

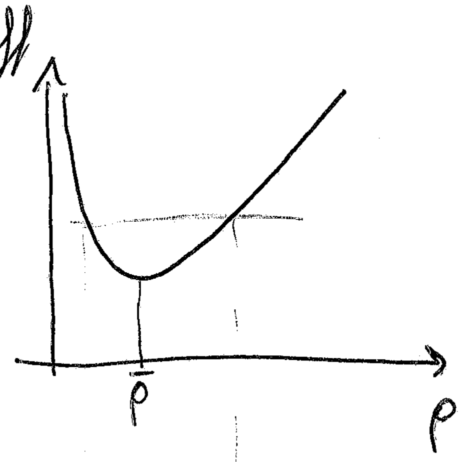
$$V_{eff} = \frac{p_\theta^2}{2m\rho^2} + \underbrace{\frac{1}{2} k \rho^8 + k\rho^6 + \left(mg - \frac{3}{2}k\right)\left(\rho^4 + \rho^2\right)}_{V_1}$$

$\alpha = mg - \frac{3}{2}k$

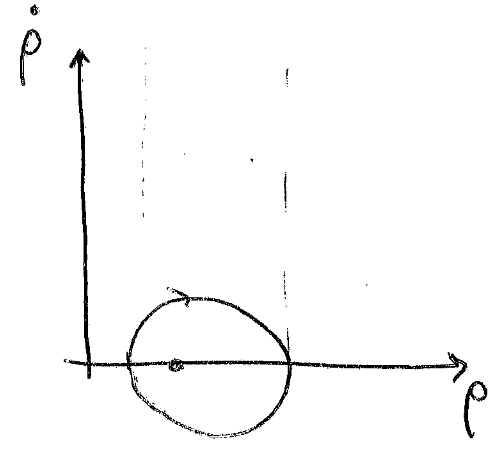
$\boxed{\alpha \geq 0}$



$\Rightarrow$



Discussione dettagliata per esercizio.

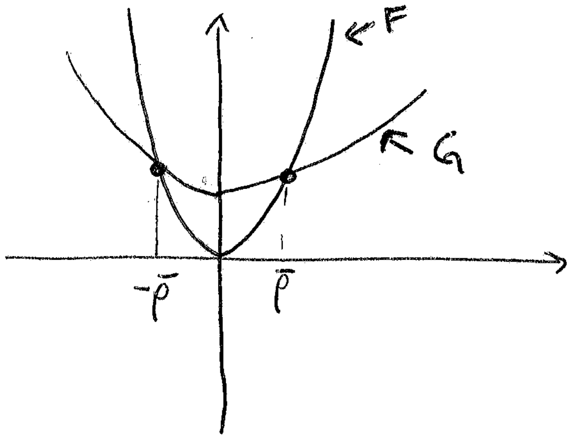


Se $\alpha < 0$

$$V'_{eff} = 4K\rho^7 + 3K\rho^5 + 4\alpha\rho^3 + 2\alpha\rho = 0$$

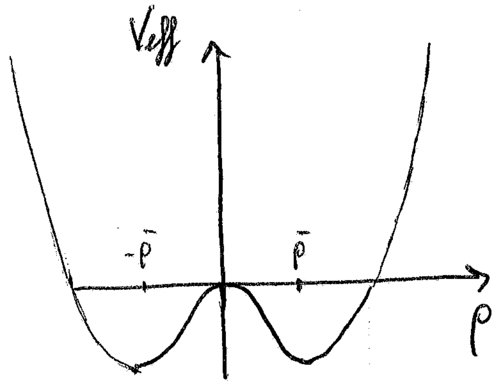
$\rho = 0$ è soluzione

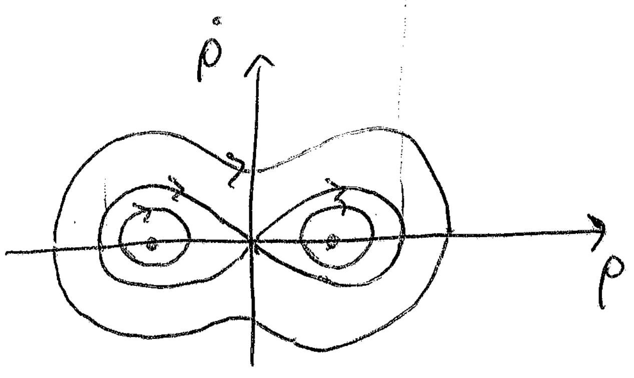
$$\underbrace{4K\rho^6 + 3K\rho^4}_{F} = \overbrace{\underbrace{-2\alpha}_{>0}\,(\rho^2+1)}^{G}$$



$\Rightarrow \bar{\rho}, -\bar{\rho}$ sono due punti di equilibrio

$P_0 = (0,0) \qquad P_1 = (\bar{\rho}, 0) \qquad P_2 = (-\bar{\rho}, 0)$





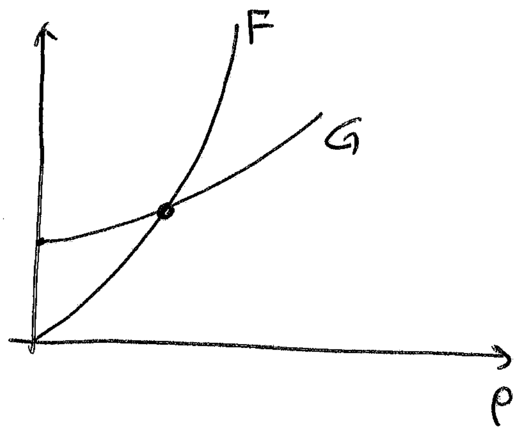
Discussione dettagliata per esercizio.

$$\boxed{\alpha < 0}$$

$$V' = -\frac{p_0^2}{m\rho^3} + 4k\rho^7 + 3k\rho^5 + 4\alpha\rho^3 + 2\alpha\rho = 0 \qquad (\rho > 0)$$

$$-\frac{p_0^2}{m} + 4k\rho^{10} + 3k\rho^8 + 4\alpha\rho^6 + 2\alpha\rho^4 = 0$$

$$\underbrace{4k\rho^{10} + 3k\rho^8}_{F} = \underbrace{\underbrace{-4\alpha}_{>0}\rho^6 \underbrace{-2\alpha}_{>0}\rho^4 + \frac{p_0^2}{m}}_{G}$$



1 sola soluzione!

È qualitativamente equivalente al caso $\alpha \geq 0$.