Traccia Soluzione (15.07.2015)

Esercizio ϕ

$$V(\underline{x}) = -e^{\sqrt{x^2+y^2}} - \frac{z^2}{2} \quad , \quad \underline{F} = -\nabla V.$$

Esercizio 1

(i) $$\frac{\partial \Phi}{\partial x} = 2x(y^2-1)(x^2y^2-1) + (x^2-1)(y^2-1)\,2xy^2$$

$$= 2x(y^2-1)(2x^2y^2 - y^2 - 1) = -\dot{y}$$

$$\frac{\partial \Phi}{\partial y} = 2y(x^2-1)(2x^2y^2 - x^2 - 1) = \dot{x}$$

(ii) $y=0 \Rightarrow x=0$ $\qquad P_0 : (0,0)$

$x=1 \Rightarrow y=\pm 1$ $\qquad P_2 : (1,1) \qquad P_2 : (1,-1)$

$x=-1 \Rightarrow y = \pm 1$ $\qquad P_3 : (-1,1) \qquad P_4 : (-1,-1)$

$$x = \pm\sqrt{\frac{1}{2y^2-1}} \Rightarrow y = \pm 1 \rightsquigarrow x = \pm 1 \Bigg\} \text{ Equilibri già determinati.}$$

$$J(x,y) = \begin{pmatrix} 8xy\left(2x^2y^2 - x^2 - y^2\right) & 2(x^2-1)\left(6x^2y^2 - x^2 - 1\right) \\ 2(1-y^2)\left(6x^2y^2 - y^2 - 1\right) & 8xy\left(2x^2y^2 - x^2 - y^2\right) \end{pmatrix}$$

$$\underbrace{J(0,0) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

CENTRO, DEVO USARE LE CURVE DI LIVELLO O LYAPUNOV.

$$\underbrace{J(\pm 1, \pm 1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$$

PUNTI DEGENERI, DEVO USARE LE CURVE DI LIVELLO.

Consideriamo la matrice Hessiana in P_0

$$H(x,y) = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial x^2} & \frac{\partial^2 \Phi}{\partial x \partial y} \\ \frac{\partial^2 \Phi}{\partial xy} & \frac{\partial^2 \Phi}{\partial y^2} \end{pmatrix} \qquad H(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

P_0 è massimo isolato, consideriamo

$$W(x,y) = H(0,0) - H(x,y) \qquad \text{(buona funzione di Lyapunov)}$$

$\Rightarrow$ P_0 è un equilibrio stabile anche per il sistema <u>non</u> lineare.

$$\Phi(P_1) = \Phi(P_2) = \Phi(P_3) = \Phi(P_4) = 0$$

studiamo quindi il luogo dei punti

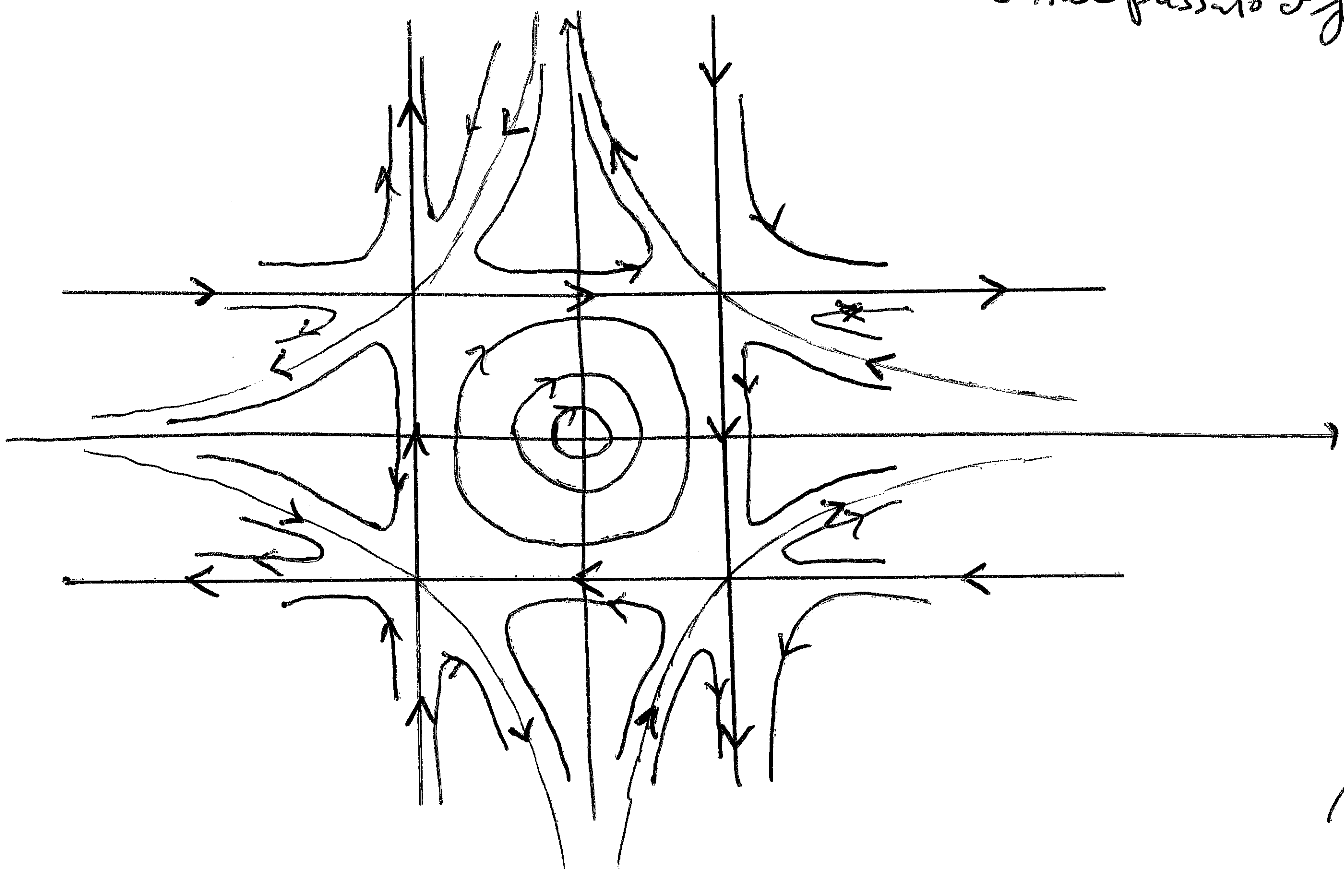
$$\Gamma_0 : \left\{ (x,y) \in \mathbb{R}^2 : (x^2-1)(y^2-1)(x^2y^2-1) = 0 \right\}$$

$$\Gamma_0 = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6$$

$C_1 : x = 1$ $\quad$ $C_3 : y = 1$ $\quad$ $C_5 : y = 1/x$

$C_2 : x = -1$ $\quad$ $C_4 : y = -1$ $\quad$ $C_6 : y = -1/x$

24 traiettorie : 4 punti equilibrio instabili
20 traiettorie asintotiche nel passato o futuro.



Esercizio 2

$$P_1 : (x_1, 0) \qquad \text{[FISSATO]}$$

$$P_2 : (R\cos\vartheta, R\sin\vartheta)$$

$$P_3 : (x_3, y_3) \qquad \text{[FISSATO]}$$

$$T = \frac{1}{2} m R^2 \dot{\vartheta}^2$$

$$V_K = \frac{K}{2}\left(\left(R\cos\vartheta - x_1\right)^2 + R^2\sin^2\vartheta + \left(R\cos\vartheta - x_3\right)^2 + \left(R\sin\vartheta - y_3\right)^2\right)$$

$$= -KR(x_1+x_3)\cos\vartheta - KRy_3\sin\vartheta \;(+\,\text{cost}).$$

$$V_g = mgR\sin\vartheta$$

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\vartheta}^2 + KR(x_1+x_3)\cos\vartheta + KRy_3\sin\vartheta - mgR\sin\vartheta$$

$$V = -KR(x_1+x_3)\cos\vartheta - R(Ky_3 - mg)\sin\vartheta$$

$$V' = KR(x_1+x_3)\sin\vartheta - R(Ky_3-mg)\cos\vartheta = 0$$

(A) $x_1 = -x_3$, $y_3 = \frac{mg}{k}$ $\Rightarrow$ $\theta \in \mathbb{T}$ è equilibrio.

(B) $x_1 = -x_3$, $y_3 \neq \frac{mg}{k}$ $\Rightarrow$ $\theta = \pm \pi/2$

(C) $x_1 \neq -x_3$, $y_3 = \frac{mg}{k}$ $\Rightarrow$ $\theta = 0, \pi$

(D) $x_1 \neq -x_3$, $y_3 \neq \frac{mg}{k}$ $\Rightarrow$ $\operatorname{tg}\theta = \dfrac{k y_3 - mg}{k(x_1 + x_3)}$

$$\left.\begin{array}{ll}
(D1) & \begin{cases} x_3 > -x_1 \\ y_3 > \frac{mg}{k} \end{cases} \\
(D2) & \begin{cases} x_3 < -x_1 \\ y_3 < \frac{mg}{k} \end{cases}
\end{array}\right\} \quad \begin{array}{l} \theta_1 \in (0, \frac{\pi}{2}) \\ \theta_2 = \theta_1 + \pi \end{array}$$

Equilibrio nel I e III quadrante

$$\left.\begin{array}{ll}
(D3) & \begin{cases} x_3 > -x_1 \\ y_3 < \frac{mg}{k} \end{cases} \\
(D4) & \begin{cases} x_3 < -x_1 \\ y_3 > \frac{mg}{k} \end{cases}
\end{array}\right\} \quad \begin{array}{l} \theta_1 \in (\pi/2, \pi) \\ \theta_2 = \theta_1 + \pi \end{array}$$

Equilibrio nel II e IV quadrante

Stabilità

$$V'' = KR(x_1 + x_3)\cos\theta + R(Ky_3 - mg)\sin\theta$$

(A) $V =$ costante , equilibrio indifferente

(B) $V''\left(\frac{\pi}{2}\right) = R(Ky_3 - mg)$

$y_3 > \frac{mg}{K}$ (STABILE) $\qquad y_3 < \frac{mg}{K}$ (INSTABILE)

$V''\left(-\frac{\pi}{2}\right) = -R(Ky_3 - mg)$

$y_3 > \frac{mg}{K}$ (INSTABILE) $\qquad y_3 > \frac{mg}{K}$ (STABILE)

(C) $V''(0) = KR(x_1 + x_3)$

$x_3 > -x_1$ (STABILE) $\qquad x_3 < -x_1$ (INSTABILE)

$V''(\pi) = -KR(x_1 + x_3)$

$x_3 > -x_1$ (INSTABILE) $\qquad x_3 < -x_1$ (STABILE)

(D) $V' = \cos(\theta)\left(KR(x_1+x_3) + R(Ky_3-mg)\dfrac{Ky_3-mg}{K(x_1+x_3)}\right)$

$$= \frac{\cos\theta\, R}{K(x_1+x_3)}\underbrace{\left(K^2(x_1+x_3)^2 + (Ky_3-mg)^2\right)}_{>0}$$

(D1) θ_1 (STABILE), θ_2 (INSTABILE)

(D2) θ_1 (INSTABILE), θ_2 (STABILE)

(D3) θ_1 (INSTABILE), θ_2 (STABILE)

(D4) θ_1 (STABILE), θ_2 (INSTABILE)

Piccole oscillazioni

$x_3 = -x_1$, $y_3 = 0$ $\Rightarrow$ $\theta = \pm \pi/2$

$\theta = -\pi/2$ STABILE

$$\left.\begin{array}{l} V'' = Rmg \\ T = \frac{1}{2} m R^2 \dot{\theta}^2 \end{array}\right\} \qquad \omega = \sqrt{\frac{g}{R}}$$

Esercizio 3

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = -e^{-\rho^2} \end{cases} \qquad \begin{cases} \dot{x} = \dot{\rho}\cos\vartheta - \rho\dot{\vartheta}\sin\vartheta \\ \dot{y} = \dot{\rho}\sin\vartheta + \rho\dot{\vartheta}\cos\vartheta \\ \dot{z} = 2\rho\dot{\rho}\, e^{-\rho^2} \end{cases}$$

$$T = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2 + 4\rho^2\dot{\rho}^2 e^{-2\rho^2} \right)$$

$$V_K = \frac{K}{2}\left(\rho^2 + e^{-2\rho^2} \right)$$

$$V_g = -mg\, e^{-\rho^2}$$

$$L = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2 + 4\rho^2\dot{\rho}^2 e^{-2\rho^2} \right) - \frac{K}{2}\left(\rho^2 + e^{-2\rho^2} \right) + mg\, e^{-\rho^2}$$

ϑ variabile ciclica $\Rightarrow$ $p_\vartheta = m\rho^2\dot{\vartheta}$ costante del moto.

$$V_{eff}(\rho) = \frac{p_\vartheta^2}{2m\rho^2} + \frac{K}{2}\rho^2 + \frac{K}{2} e^{-2\rho^2} - mg\, e^{-\rho^2}$$

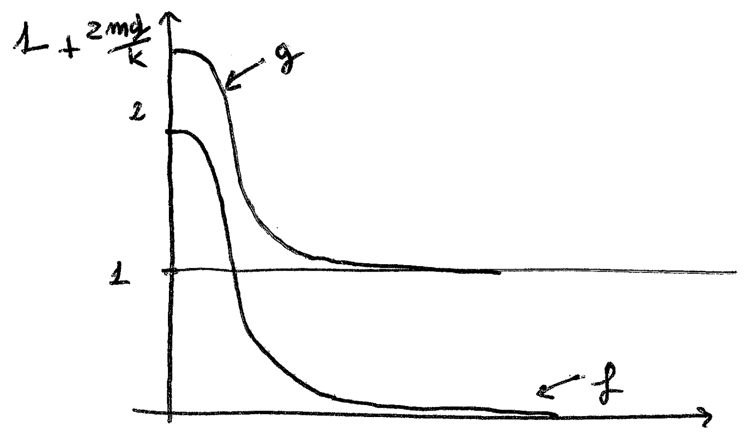
$\boxed{\rho_\theta = \emptyset} \implies \rho \in \mathbb{R}$

$$V'(\rho) = k\rho - 2k\rho e^{-2\rho^2} + 2mg\rho e^{-\rho^2}$$

$$= k\rho\left(1 - 2e^{-2\rho^2} + 2\frac{mg}{k}e^{-\rho^2}\right) = 0$$

$\rho = 0$ $\qquad\qquad 2e^{-2\rho^2} - 2\frac{mg}{k}e^{-\rho^2} - 1 = 0$

$$\underbrace{2e^{-2\rho^2}}_{f} = \underbrace{1 + \frac{2mg}{k}e^{-\rho^2}}_{g}$$

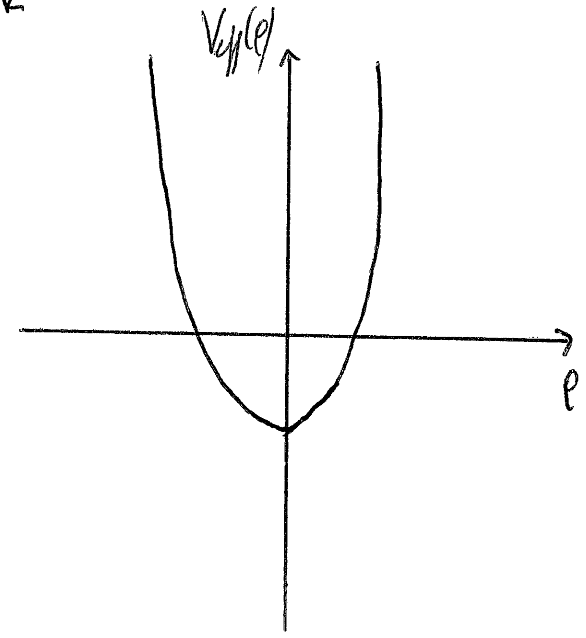


Una sola intersezione e solo se $\frac{2mg}{k} \leq 1$.

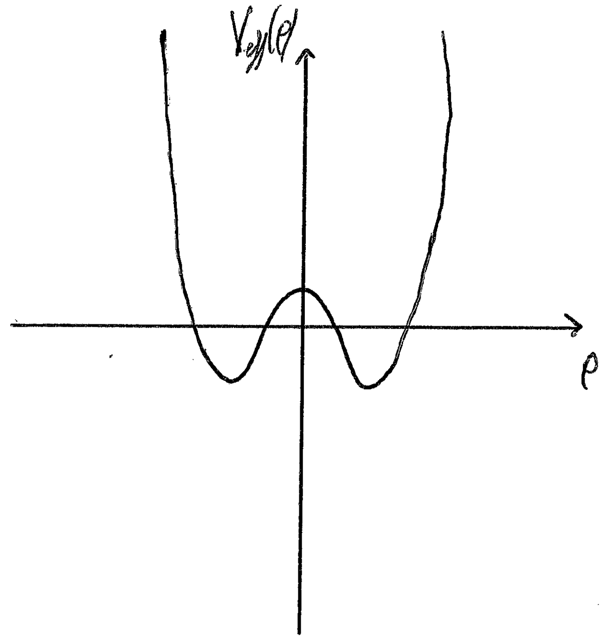
⚠ Si poteva anche risolvere esplicitamente l'equazione ottenendo

$$\rho_{1,2} = \pm\sqrt{-\ln\left(\frac{\frac{2mg}{k} + \sqrt{\left(\frac{2mg}{k}\right)^2 + 8}}{4}\right)}$$

$$\frac{2mg}{k} \geq 1$$



$$\frac{2mg}{k} < 1$$



... discussione e ritratto di fase per esercizio...

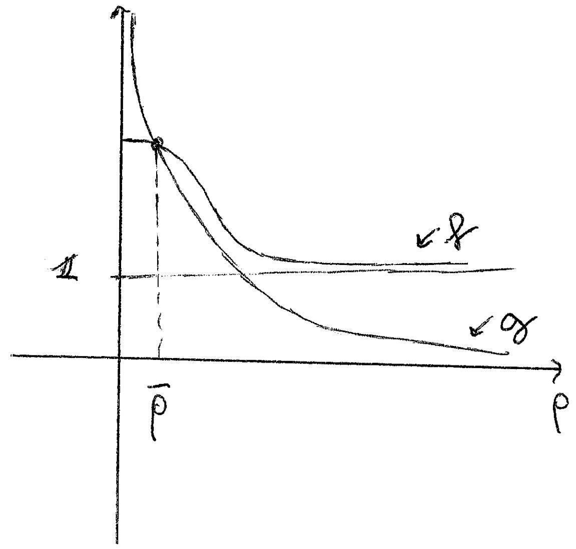
$\boxed{p_\theta \neq 0} \Rightarrow \rho > 0$

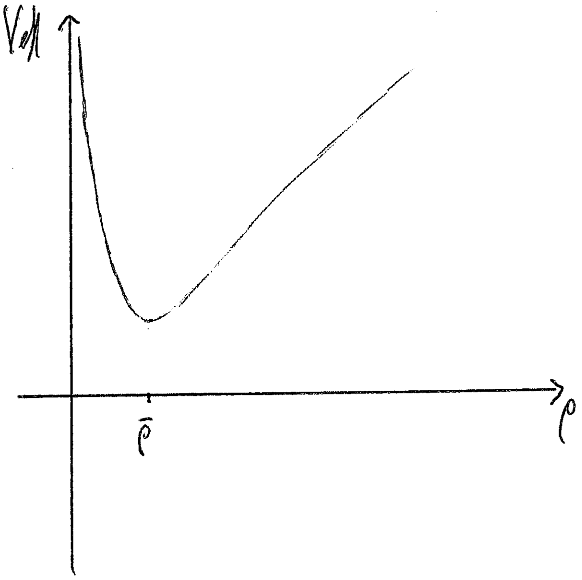
$$V'_{eff}(\rho) = -\frac{p_\theta^2}{m\rho^3} + k\rho - 2k\rho\, e^{-2\rho^2} + 2mg\rho\, e^{-\rho^2} = 0$$

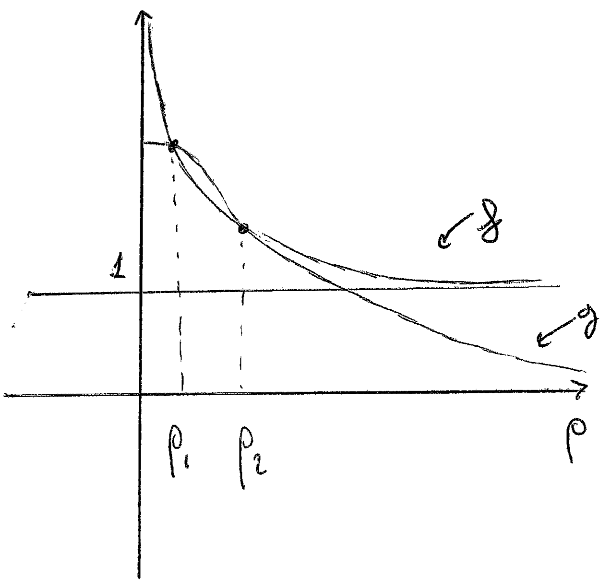
In questo caso $\rho \neq 0$ $(\rho > 0)$ quindi otteniamo

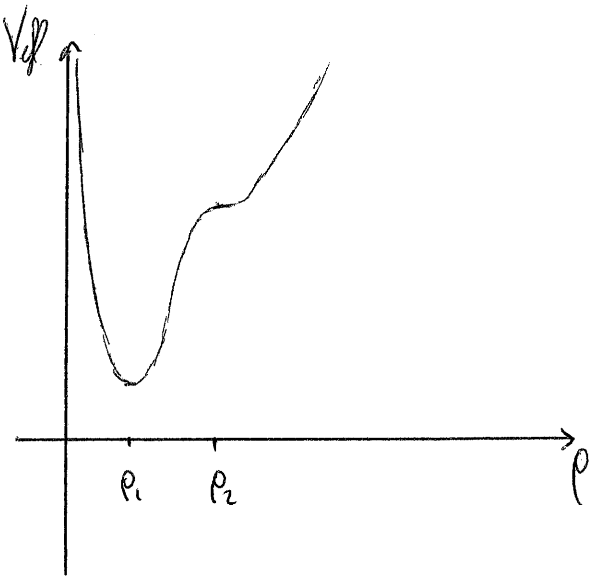
$$\underbrace{\left(1 + 2\frac{mg}{k} e^{-\rho^2}\right)}_{f} = \underbrace{\left(2e^{-2\rho^2} + \frac{p_\theta^2}{mk\rho^4}\right)}_{g}$$

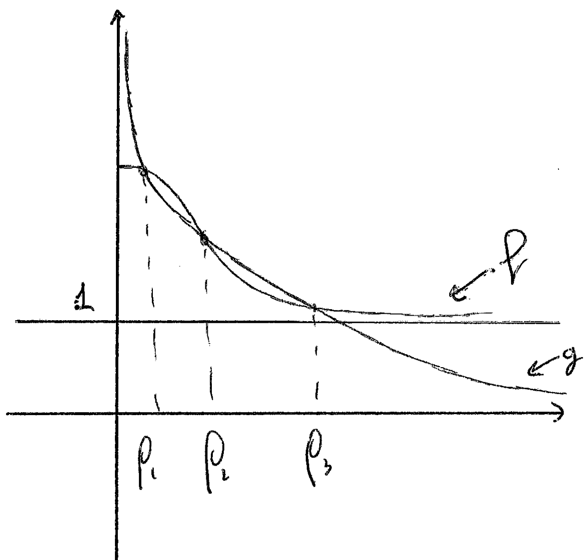
Lo studio analitico non è assolutamente banale, ma dal punto di vista qualitativo (che è quel che ci interessa) distinguiamo tre casi possibili

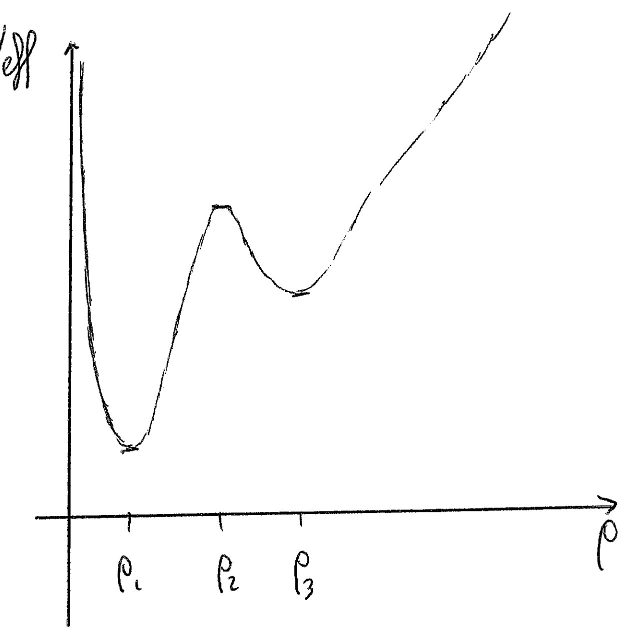












... ritratti di fase per esercizio..

⚠ L'esercizio 3 è stato valutato complessivamente 10 punti. Considerando <u>solo</u> il caso $p_0 = 0$ si poteva arrivare ad un massimo di 8 punti.

⚠ Lo studio dettagliato del caso $p_0 \neq 0$ era la parte più complessa del compito, ma <u>non</u> influiva in maniera pesante nella valutazione complessiva.