Esercizio ϕ

$$\dot{x} = yz \qquad \dot{y} = -xz \qquad \dot{z} = xy$$

$\Phi(x, y, z)$ costante del moto $\Leftrightarrow \frac{d}{dt}\Phi = \phi$

$$\frac{d}{dt}\Phi = \frac{\partial \Phi}{\partial x}\dot{x} + \frac{\partial \Phi}{\partial y}\dot{y} + \frac{\partial \Phi}{\partial z}\dot{z}$$

$$= \frac{\partial \Phi}{\partial x}(yz) + \frac{\partial \Phi}{\partial y}(-xz) + \frac{\partial \Phi}{\partial z}(xy)$$

$$\Phi_1(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2}$$

$$\Phi_2(x, y, z) = \frac{y^2}{2} + \frac{z^2}{2}$$

Φ_1 e Φ_2 sono funzionalmente indipendenti

Esercizio 1

$$\begin{cases} \dot{x} = (y-x)(1-x-y) \\ \dot{y} = x(2+y) \end{cases}$$

i) $P_0 : (0,0)$ $\quad P_1 : (0,1)$

$P_2 : (-2,-2)$ $\quad P_3 : (3,-2)$

$$J(x,y) = \begin{pmatrix} 2x-1 & 1-2y \\ 2+y & x \end{pmatrix}$$

$$J(P_0) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

$(-1-\lambda)(-\lambda)-2=0$

$\lambda^2+\lambda-2=0$

$\boxed{\lambda_1 = 1}$ $\quad \boxed{\lambda_2 = -2}$

SELLA

$$J(P_1) = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}$$

$(-1-\lambda)(-\lambda)+3=0$

$\lambda^2+\lambda+3=0$

$\boxed{\lambda_{1,2} = \frac{-1 \pm i\sqrt{11}}{2}}$

FUOCO STABILE

$$J(P_2) = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix}$$ NODO STABILE

$$J(P_3) = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}$$ NODO INSTABILE

ii)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda_1 = 1 \qquad \lambda_2 = -2$$

$$v^{(1)} : \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_x^{(1)} \\ v_y^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$v^{(2)} : \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_x^{(2)} \\ v_y^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \qquad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\exp(tA) = P \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{pmatrix} P^{-1} =$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{t} & e^{t} \\ 2e^{-2t} & -e^{-2t} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^{t}+2e^{-2t} & e^{t}-e^{-2t} \\ 2e^{t}-2e^{-2t} & 2e^{t}+e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^{t}+2e^{-2t} & e^{t}-e^{-2t} \\ 2e^{2t}-2e^{-2t} & 2e^{t}+e^{-2t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

iii)
$$\begin{cases} \tilde{x} = x \\ \tilde{y} = y - 1 \end{cases} \qquad \begin{cases} \dot{\tilde{x}} = \dot{x} \\ \dot{\tilde{y}} = \dot{y} \end{cases}$$

$$\begin{cases} \dot{\tilde{x}} = (\tilde{y} + 1 - x)(-\tilde{x} - \tilde{y}) \\ \dot{\tilde{y}} = \tilde{x}(\tilde{y} + 3) \end{cases}$$

$$\Phi = \frac{1}{2}\left(a\tilde{x}^2 + b\tilde{y}^2\right)$$

$$\dot{\Phi} = a\tilde{x}(\tilde{y} - \tilde{x} + 1)(-\tilde{x} - \tilde{y}) + b\tilde{y}\,\tilde{x}(\tilde{y} + 3)$$

$$= a\tilde{x}\left(-\tilde{x} - \tilde{y} + \tilde{x}^2 - \tilde{y}^2\right) + b\,\tilde{x}\tilde{y}^2 + 3b\,\tilde{x}\tilde{y}$$

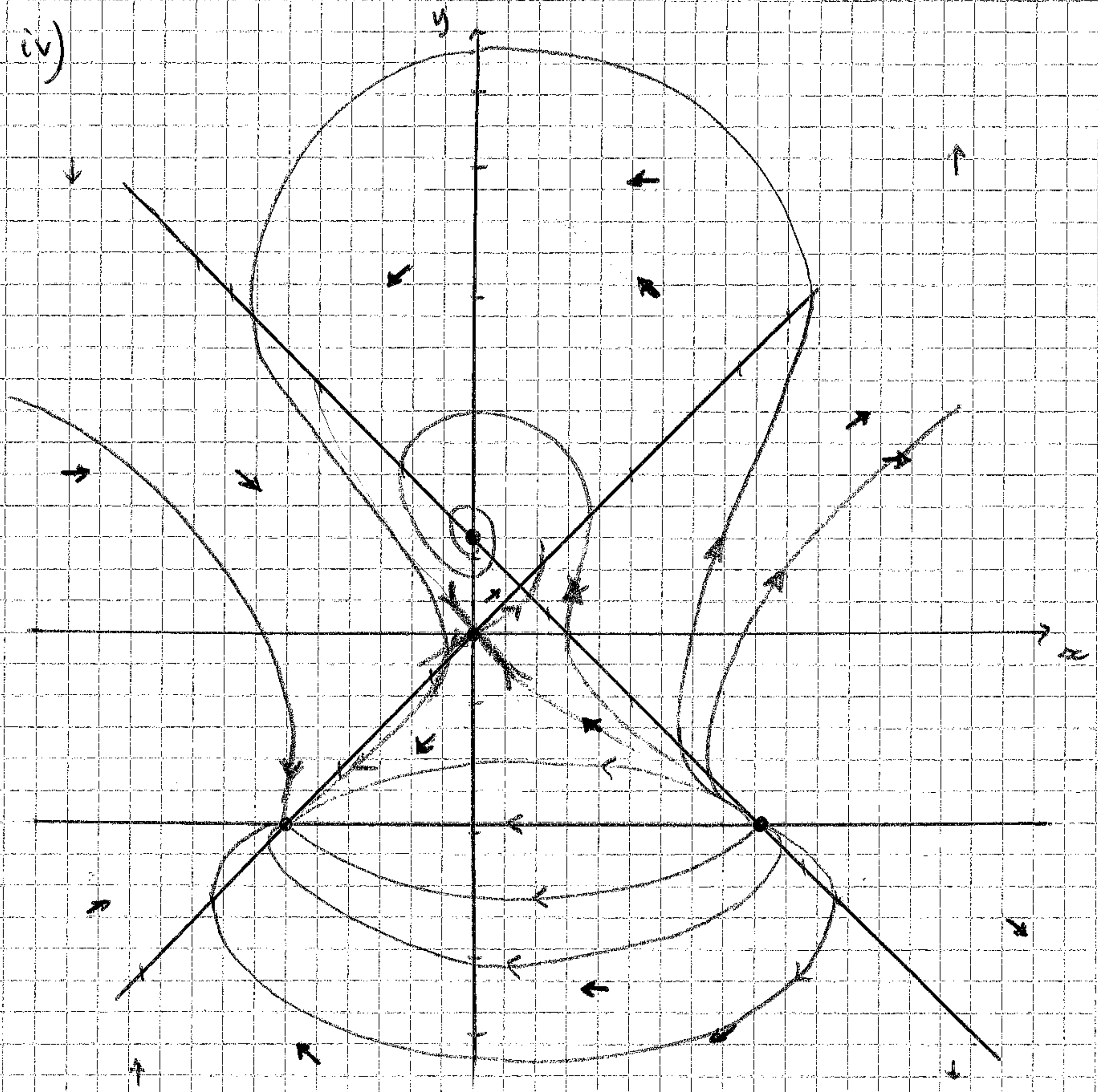
$$= -a\tilde{x}^2 + a\tilde{x}\tilde{y} + a\tilde{x}^3 - a\tilde{x}\tilde{y}^2 + b\tilde{x}\tilde{y}^2 + 3b\tilde{x}\tilde{y}$$

$\boxed{a = 3,\ b = 1}$ $\Rightarrow (a + 3b)\tilde{x}\tilde{y} = 0$ ✓

$$= -3\tilde{x}^2 + 3\tilde{x}^3 - 2\tilde{x}\tilde{y}^2 \leq 0$$

In un intorno non ci sono altri equilibri, ma basta Lyapunov debole.

iv)



Esercizio 2

i) Coordinate

$$\begin{cases} P_1 = \left(s_1 \sin(\gamma),\ -s_1 \cos(\gamma)\right) \\ P_2 = \left((s_1+s_2)\sin(\gamma),\ -(s_1+s_2)\cos(\gamma)\right) \end{cases}$$

$$\begin{cases} \dot{P}_1 = \left(\dot{s}_1 \sin(\gamma),\ -\dot{s}_1 \cos(\gamma)\right) \\ \dot{P}_2 = \left((\dot{s}_1+\dot{s}_2)\sin(\gamma),\ -(\dot{s}_1+\dot{s}_2)\cos(\gamma)\right) \end{cases}$$

$$T = \frac{1}{2} m \left(2\dot{s}_1^2 + \dot{s}_2^2 + 2\dot{s}_1\dot{s}_2\right)$$

$$V = -2mg\, s_1 \cos(\gamma) - mg\, s_2 \cos(\gamma) + \frac{1}{2} K s_1^2 \cos^2(\gamma) + \frac{1}{2} K s_2^2$$

$$L = T - V$$

ii) <u>Equilibri</u>

$$\begin{cases} \dfrac{\partial V}{\partial s_1} = -2mg\cos(\gamma) + K s_1 \cos^2(\gamma) = 0 \\ \\ \dfrac{\partial V}{\partial s_2} = -mg\cos(\gamma) + K s_2 = 0 \end{cases}$$

$$\bar{s}_1 = \frac{2mg}{K\cos(\gamma)} \qquad \bar{s}_2 = \frac{mg\cos(\gamma)}{K}$$

$$\left(s_1 = \bar{s}_1 \quad s_2 = \bar{s}_2 \quad \dot{s}_1 = 0 \quad \dot{s}_2 = 0 \right)$$

Stabilità

$$\frac{\partial^2 V}{\partial s_1^2} = K\cos^2(\gamma), \quad \frac{\partial^2 V}{\partial s_1 \partial s_2} = 0, \quad \frac{\partial^2 V}{\partial s_2^2} = K$$

$\Rightarrow$ $\boxed{\text{Equilibrio stabile}}$

iii) $\gamma = +\frac{\pi}{4}$

$$A = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$$

$$B = \begin{pmatrix} K\cos^2\gamma & 0 \\ 0 & K \end{pmatrix}$$

$$\det\left(B - \lambda A\right) = 0$$

$$2m^2\lambda^2 - \left(2Km + mK\cos^2(\gamma)\right)\lambda + K^2\cos^2\gamma - \lambda^2 m^2 = 0$$

$$m^2\lambda^2 - \left(2Km + mK\cos^2(\gamma)\right)\lambda + K^2\cos^2\gamma = 0$$

$\gamma = \pi/4$

$$m^2\lambda^2 - \frac{5}{2}Km\,\lambda + \frac{K^2}{2} = 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{17}\,K}{4m}$$

$$u^{(1)} = \begin{pmatrix} 1 \\ -\frac{3+\sqrt{17}}{4} \end{pmatrix} \qquad u^{(2)} = \begin{pmatrix} 1 \\ -\frac{3-\sqrt{17}}{4} \end{pmatrix}$$

Esercizio 3

$$P_1 : \left(l \sin\vartheta, \ -l\cos\vartheta\right)$$

$$P_2 \ \left(-l\sin\vartheta, \ -l\cos\vartheta\right)$$

$$\xi_1 = l\sin\vartheta\cos\omega t \qquad \eta_1 = l\sin\vartheta\sin\omega t$$

$$\xi_2 = -l\sin\vartheta\cos\omega t \qquad \eta_2 = -l\sin\vartheta\sin\omega t$$

$$\dot{\xi}_1 = l\dot{\vartheta}\cancel{\cos}\cos\vartheta\cos\omega t - \omega l\sin\vartheta\sin\omega t$$

$$\dot{\xi}_2 = -l\dot{\vartheta}\cos\vartheta\cos\omega t + \omega l\sin\vartheta\sin\omega t$$

$$\dot{\eta}_1 = l\dot{\vartheta}\cos\vartheta\sin\omega t + \omega l\sin\vartheta\cos\omega t$$

$$\dot{\eta}_2 = -l\dot{\vartheta}\cos\vartheta\sin\omega t - \omega l\sin\vartheta\cos\omega t$$

$$T = \frac{1}{2}m\left(\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2}m\left(2\,l^2\dot{\vartheta}^2\cos^2\vartheta + 2\omega^2 l^2\sin^2\vartheta + 2l^2\dot{\vartheta}^2\sin^2\vartheta\right)$$

$$= \frac{1}{2}m\left(2l^2\dot{\vartheta}^2 + 2\omega^2 l^2\sin^2\vartheta\right)$$

$$= ml^2\dot{\vartheta}^2 + m\omega^2 l^2\sin^2\vartheta$$

$$V = -2mgl\cos\vartheta$$

$$L = ml^2\dot{\vartheta}^2 + \underbrace{m\omega^2 l^2 \sin^2\vartheta + 2mgl\cos\vartheta}_{\tilde{V}}$$

$$\tilde{V} = -2mgl\cos\vartheta - m\omega^2 l^2 \sin^2\vartheta$$

$$\tilde{V}' = 2mgl\sin\vartheta - 2m\omega^2 l^2 \sin\vartheta\cos\vartheta = 0$$

$$\sin\vartheta = 0 \Rightarrow \vartheta = 0, \pi$$

$$\cos\vartheta = \frac{mgl}{m\omega^2 l^2} = \frac{g}{l}\cdot\frac{1}{\omega^2} > 0$$

$$\left(\frac{1}{\omega^2}\cdot\frac{g}{l}\right) > 1 \Rightarrow \vartheta_{1,2} = 0, \pi \quad \boxed{\text{2 EQUILIBRI}}$$

$$\left(\frac{1}{\omega^2}\cdot\frac{g}{l}\right) \leq 1 \Rightarrow \vartheta_{1,2} = 0, \pi$$

$$\vartheta_{3,4} = \pm\, a\cos\left(\frac{1}{\omega^2}\cdot\frac{g}{l}\right)$$

(LI IDENTIFICO)

$$\boxed{\text{3 EQUILIBRI}}$$

$$\tilde{V}'' = 2mgl\cos\vartheta - 2m\omega^2 l^2 \cos(2\vartheta)$$

$\underline{\theta = \pi}$

$$-2mgl - 2m\omega^2 l^2 < 0 \qquad \text{INSTABILE}$$

$\underline{\theta = 0}$

$$2mgl - 2m\omega^2 l^2 < 0$$

$$\Updownarrow$$

$$2mgl < 2m\omega^2 l^2$$

$$\Updownarrow$$

$$\left(\frac{1}{\omega^2}\frac{g}{l}\right) < 1 \qquad \text{INSTABILE}$$

$$\left(\frac{1}{\omega^2}\frac{g}{l}\right) \geq 1 \qquad \text{STABILE}$$

$\underline{\theta = \theta^*}$ $\left(\text{SE } \left(\frac{1}{\omega^2}\, g/l\right) < 1\right)$

$$2mgl\left(\frac{1}{\omega^2}\frac{g}{l}\right) - 2m\omega^2 l^2\left(\left(\frac{1}{\omega^2}\frac{g}{l}\right)^2 - \sin^2(\theta^*)\right) =$$

$$= \cancel{\frac{2mg^2}{\omega^2}} - \cancel{\frac{2mg^2}{\omega^2}} + 2m\omega^2 l^2 \sin^2(\theta^*) > 0$$

$\theta = \theta^*$ STABILE