Compito (20 Novembre 2015)

Esercizio ϕ

$$\underline{F} = \left(xy - \sin z \,,\ \frac{x^2}{2} - \frac{e^y}{z} \,,\ \frac{e^y}{z^2} - x \cos z \right)$$

nel dominio $D = \{ (x,y,z) \in \mathbb{R}^3 : z > 0 \}$

$$\frac{\partial F_x}{\partial y} = x \qquad \frac{\partial F_y}{\partial x} = x \qquad \boxed{\checkmark}$$

$$\frac{\partial F_x}{\partial z} = -\cos z \qquad \frac{\partial F_z}{\partial x} = -\cos z \qquad \boxed{\checkmark}$$

$$\frac{\partial F_y}{\partial z} = \frac{e^y}{z^2} \qquad \frac{\partial F_z}{\partial y} = \frac{e^y}{z^2} \qquad \boxed{\checkmark}$$

D semplicemente connesso $\Rightarrow$ $\underline{F}$ conservativa.

$$V = \int xy - \sin z \, dx = \frac{x^2 y}{2} - x \sin z + \varphi(y,z)$$

$$\frac{\partial V}{\partial y} = \frac{x^2}{2} + \frac{\partial \varphi}{\partial y} \overset{def.}{=} \frac{x^2}{2} - \frac{e^y}{z} \Rightarrow \underline{\varphi(y,z) = -\frac{e^y}{z} + \psi(z)}$$

$$\frac{\partial V}{\partial z} = -x \cos z + \frac{e^y}{z^2} + \frac{d\psi}{dz} \overset{def}{=} -x \cos z + \frac{e^y}{z^2} \Rightarrow \underline{\psi(z) = \phi}$$

$$\boxed{U = -V = -\frac{x^2 y}{2} + x \sin z + \frac{e^y}{z}}$$

Esercizio 1

$$\begin{cases} \dot{x} = 2y - (x^2-1)^2 - 1 & = \dfrac{\partial \Phi}{\partial y} \\ \dot{y} = 4x(x^2-1)(y-1) & = -\dfrac{\partial \Phi}{\partial x} \end{cases}$$

i) Dalla prima equazione

$$\Phi(x,y) = y^2 - \left((x^2-1)^2 - 1\right)y + \varphi(x)$$

$$\frac{\partial \Phi}{\partial x} = -4x(x^2-1)y + \frac{d\varphi}{dx} \overset{\text{def}}{=} -4x(x^2-1)(y-1)$$

$$\Rightarrow \frac{d\varphi}{dx} = 4x(x^2-1) \quad \Rightarrow \quad \underline{\varphi(x) = x^4 - 2x^2}$$

$$\boxed{\Phi = y^2 - \left((x^2-1)^2 - 1\right)y + x^2(x^2-2)}$$

ii)

$$\begin{cases} 2y - (x^2-1)^2 - 1 = 0 \\ 4x(x^2-1)(y-1) = 0 \end{cases}$$

$P_1 : (0, 1)$ $P_2 : (\sqrt{2}, 1)$ $P_3 : (-\sqrt{2}, 1)$

$P_4 : (1, 1/2)$ $P_5 : (-1, 1/2)$

Calcolo della Jacobiana

$$J(x,y) = \begin{pmatrix} -4x(x^2-1) & 2 \\ 4(3x^2-1)(y-1) & 4x(x^2-1) \end{pmatrix}$$

$$J(P_1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$ autovalori nulli

LA LINEARIZZAZIONE NON BASTA.

$$J(P_2/P_3) = \begin{pmatrix} \mp 4\sqrt{2} & 2 \\ 0 & \pm 4\sqrt{2} \end{pmatrix}$$ SELLA INSTABILE

$$J(P_4/P_5) = \begin{pmatrix} 0 & 2 \\ -4 & 0 \end{pmatrix}$$ $\lambda = \pm 2\sqrt{2}\, i$

LINEARIZZAZIONE NON ATTENDIBILE

Consideriamo l'Hessiana in P_4 e P_5

$$\mathcal{H}(P_4/P_5) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$ P_4 e P_5 MINIMI

$$W(x,y) = \Phi(x,y) - \Phi(P_4/P_5)$$ buone funzioni Lyapunov

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P_4 e P_5 sono equilibri <u>STABILI</u>. (CENTRI)

P_2 e P_3 sono equilibri <u>INSTABILI</u> (SELLE)

P_1 da determinarsi dalle curve di livello di Φ.

$$\Phi(P_1) = \Phi(P_2) = \Phi(P_3) = -1$$

Consideriamo quindi la curva di livello

$$\Gamma_1 = \{(x,y) \in \mathbb{R}^2 : \Phi(x,y) = -1\}$$

$$y^2 - \left((x^2-1)^2+1\right)y + x^2(x^2-2)+1 = 0$$

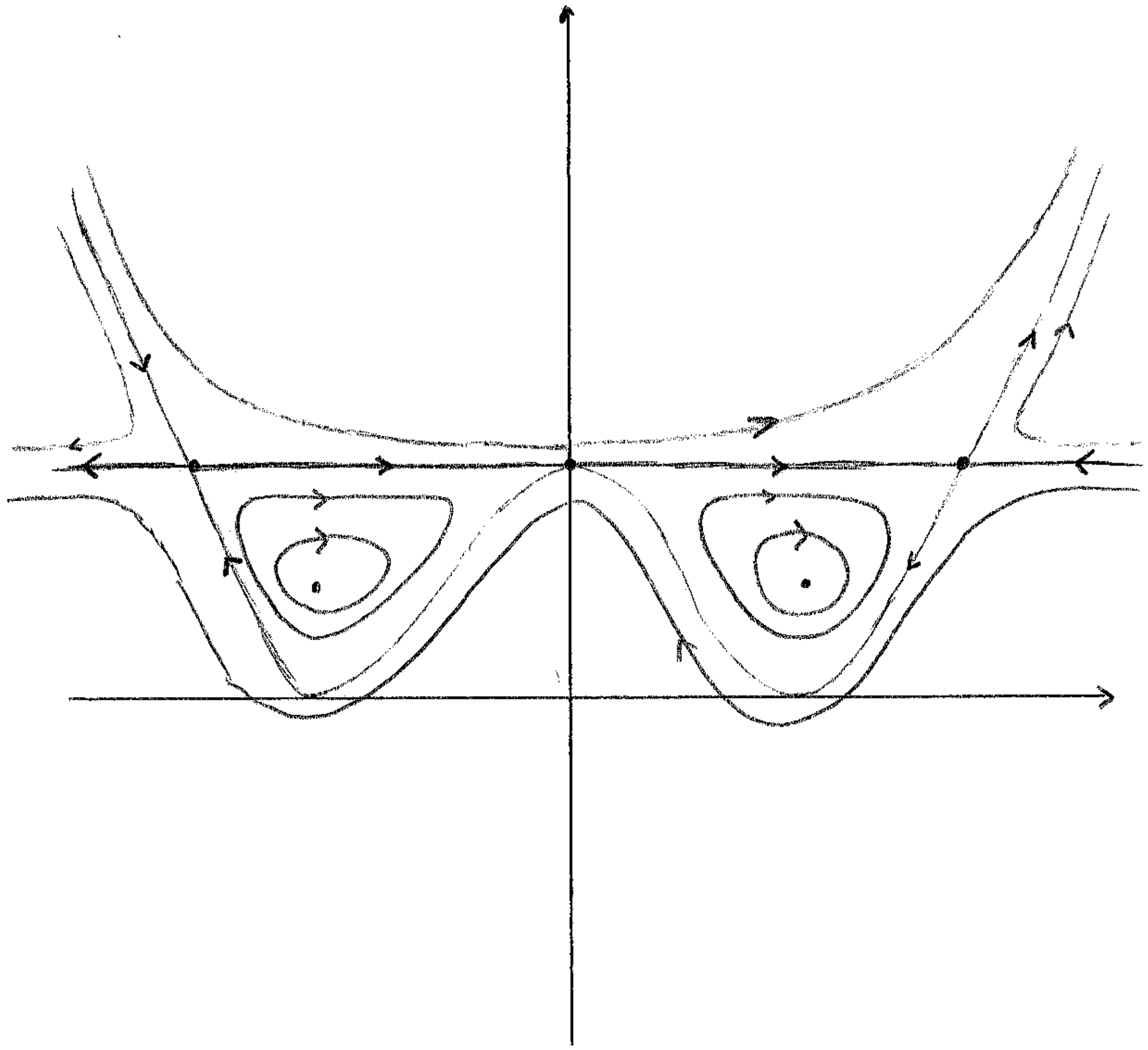
$$y^2 - y - (x^2-1)^2 y + (x^2-1)^2 = 0$$

$$(y-1)\left(y-(x^2-1)^2\right) = 0$$

$$\Gamma_1 = C_1 \cup C_2$$

$$C_1 : \left\{(x,y)\in\mathbb{R}^2 : y=1\right\}$$

$$C_2 : \left\{(x,y)\in\mathbb{R}^2 : y=(x^2-1)^2\right\}$$



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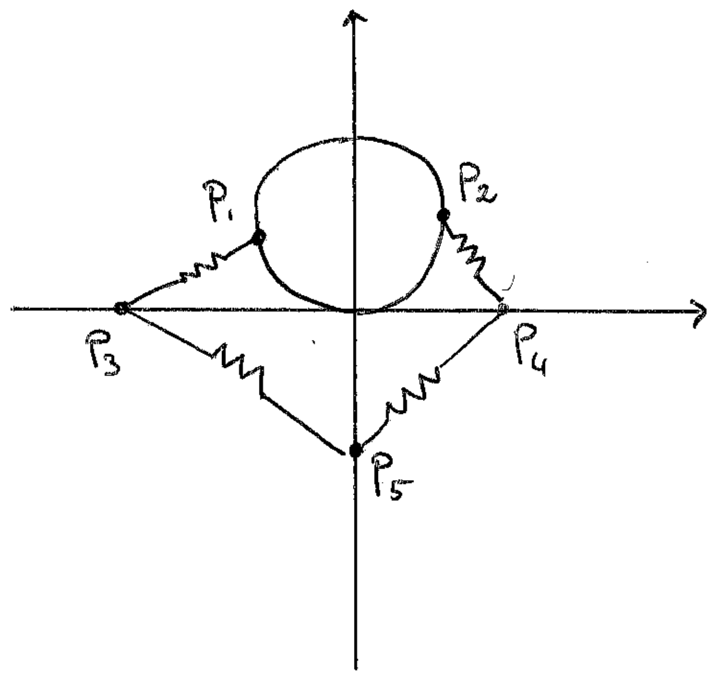
Esercizio 2



$P_1 : (\sin\vartheta_1 , 1 - \cos\vartheta_1)$ $\qquad P_3 = (x_1 , 0)$ $\qquad P_5 = (0, y)$

$P_2 : (\sin\vartheta_2 , 1 - \cos\vartheta_2)$ $\qquad P_4 = (x_2 , 0)$

$$T = \frac{m}{2}\left(\dot\vartheta_1^2 + \dot\vartheta_2^2 + \dot x_1^2 + \dot x_2^2 + \dot y^2 \right)$$

$$V = mg\left(1 - \cos\vartheta_1 + 1 - \cos\vartheta_2 + y \right)$$
$$+ \frac{K}{2}\Big((x_1 - \sin\vartheta_1)^2 + (x_2 - \sin\vartheta_2)^2 + (1 - \cos\vartheta_1)^2 + (1 - \cos\vartheta_2)^2$$
$$+ x_1^2 + x_2^2 + 2y^2 \Big)$$

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ii) Equilibri $(\nabla V = 0)$

$$\begin{cases} mg\sin\vartheta_1 - K x_1\cos\vartheta_1 + K\sin\vartheta_1 = 0 \\ mg\sin\vartheta_2 - K x_2\cos\vartheta_2 + K\sin\vartheta_2 = 0 \\ 2K x_1 - K\sin\vartheta_1 = 0 \\ 2K x_2 - K\sin\vartheta_2 = 0 \\ 2Ky + mg = 0 \end{cases}$$

$$\bar{y} = -\frac{mg}{2K} \qquad 2x_1 = \sin\vartheta_1 \qquad 2x_2 = \sin\vartheta_2$$

$$\frac{\sin\vartheta_1}{2}\left(2(mg+K) - K\cos\vartheta_1\right) = 0 \quad \Rightarrow \bar{\vartheta}_1 = 0, \ \bar{\vartheta}_1 = \pi$$

Analogamente $\bar{\vartheta}_2 = 0$, $\bar{\vartheta}_2 = \pi$

Otteniamo quattro configurazioni di equilibrio

(A) $\vartheta_1 = 0$ $\quad \vartheta_2 = 0$ $\quad x_1 = 0$ $\quad x_2 = 0$ $\quad y = -\frac{mg}{2K}$

(B) $\vartheta_1 = 0$ $\quad \vartheta_2 = \pi$ $\quad x_1 = 0$ $\quad x_2 = 0$ $\quad y = -\frac{mg}{2K}$

(C) $\vartheta_1 = \pi$ $\quad \vartheta_2 = 0$ $\quad x_1 = 0$ $\quad x_2 = 0$ $\quad y = -\frac{mg}{2K}$

(D) $\vartheta_1 = \pi$ $\quad \vartheta_2 = \pi$ $\quad x_1 = 0$ $\quad x_2 = 0$ $\quad y = -\frac{mg}{2K}$

Trascurando le costanti, possiamo riscrivere

$$V = mg\left(y - \cos\vartheta_1 - \cos\vartheta_2\right)$$
$$+K\left(x_1^2 + x_2^2 + y^2 - x_1 \sin\vartheta_1 - \cos\vartheta_1 - x_2 \sin\vartheta_2 - \cos\vartheta_2\right)$$

$$\mathcal{L} = T - V$$

$$\ddot{\vartheta}_1 = -g\sin\vartheta_1 + \frac{K}{m} x_1 \cos\vartheta_1 - \frac{K}{m}\sin\vartheta_1$$

$$\ddot{\vartheta}_2 = -g\sin\vartheta_2 + \frac{K}{m} x_2 \cos\vartheta_2 - \frac{K}{m}\sin\vartheta_2$$

$$\ddot{x}_1 = -2\frac{K}{m} x_1 + \frac{K}{m}\sin\vartheta_1$$

$$\ddot{x}_2 = -2\frac{K}{m} x_2 + \frac{K}{m}\sin\vartheta_2$$

$$\ddot{y} = -2Ky - g$$

Per discutere la stabilità osserviamo che la Lagrangiana si separa nella somma di <u>tre</u> Lagrangiane indipendenti

$$L = L_1(\vartheta_1, x_1, \dot\vartheta_1, \dot x_1) + L_2(\vartheta_2, x_2, \dot\vartheta_2, \dot x_2) + L_3(y, \dot y)$$

$$L_1 = \frac{m}{2}\left(\dot\vartheta_1^2 + \dot x_1^2\right) + mg\cos\vartheta_1 - K\left(x_1^2 - x_1\sin\vartheta_1 - \cos\vartheta_1\right)$$

$$L_2 = \frac{m}{2}\left(\dot\vartheta_2^2 + \dot x_2^2\right) + mg\cos\vartheta_2 - K\left(x_2^2 - x_2\sin\vartheta_2 - \cos\vartheta_2\right)$$

$$L_3 = \frac{m}{2}\dot y^2 - mgy - Ky^2$$

$$L_1 = T_1 - V_1 \quad , \quad L_2 = T_2 - V_2 \quad , \quad L_3 = T_3 - V_3$$

Consideriamo L_3,

$$\frac{\partial^2 V_3}{\partial y^2} = 2K > 0 \quad \Rightarrow \quad y = -\frac{mg}{2K} \quad \text{minimo}$$

$$(y, \dot y) = \left(-\frac{mg}{2K}, 0\right) \quad \text{EQUILIBRIO STABILE}$$

Consideriamo L_1 (per L_2 vale lo stesso)

$$\frac{\partial^2 V_1}{\partial \vartheta_1^2} = mg\cos\vartheta_1 + K\cos\vartheta_1 + K x_1 \sin\vartheta_1$$

$$\frac{\partial^2 V_1}{\partial \vartheta_1 \partial x_1} = -K\cos\vartheta_1$$

$$\frac{\partial^2 V_1}{\partial x_1^2} = 2K$$

$$\mathcal{H}(0,0) = \begin{pmatrix} mg+K & -K \\ -K & 2K \end{pmatrix} \qquad (\vartheta_1, x_1) = (0,0)$$

<u>minimo</u>

$$\mathcal{H}(\pi,0) = \begin{pmatrix} -mg-K & -K \\ -K & 2K \end{pmatrix} \qquad (\vartheta_1, x_1) = (\pi,0)$$

<u>sella</u>

$\mathcal{H}(0,\pi)$ analogo $\qquad (\vartheta_1, x_1) = (0,\pi)$

<u>sella</u>

$\Rightarrow$ (A) stabile

(B), (C), (D) instabili

iii) Essendo P_1 e P_2 fissato, devo considerare solamente la forza centrifuga agente sui punto P_3 e P_4, introducendo il potenziale

$$\tilde{V} = -\frac{1}{2} m \omega^2 \left(x_1^2 + x_2^2 \right)$$

La Lagrangiana del sistema diventa

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}^2 \right) - \left(mgy + K\left(x_1^2 + x_2^2 + y^2 - x_1 + x_2 \right) + \tilde{V} \right)$$

che ancora posso spezzare come

$$\mathcal{L}_1 = \frac{m}{2}\dot{x}_1^2 - Kx_1^2 + Kx_1 + \frac{1}{2} m\omega^2 x_1^2$$

$$\mathcal{L}_2 = \frac{m}{2}\dot{x}_2^2 - Kx_2^2 + Kx_2 + \frac{1}{2} m\omega^2 x_2^2$$

$$\mathcal{L}_3 = \frac{m}{2}\dot{y}^2 - mgy - Ky^2$$

Ho equilibrio solo se $2K \neq m\omega^2$

(E) $\quad x_1 = \dfrac{K}{2K - m\omega^2}, \quad x_2 = -\dfrac{K}{2K - m\omega^2}, \quad y = -\dfrac{mg}{2K}$

Infatti ponendo $\nabla V = 0$ otteniamo

$$\frac{\partial V}{\partial x_1} = 2Kx_1 - K - m\omega^2 x_1 = 0$$

$$\frac{\partial V}{\partial x_2} = 2Kx_2 + K - m\omega^2 x_2 = 0$$

$$\frac{\partial V}{\partial y} = 2Ky + my = 0$$

Calcoliamo la matrice Hessiana

$$\left.\begin{aligned} \frac{\partial^2 V}{\partial x_1^2} &= 2K - m\omega^2 \\ \frac{\partial^2 V}{\partial x_2^2} &= 2K - m\omega^2 \\ \frac{\partial^2 V}{\partial y^2} &= 2K > 0 \end{aligned}\right\}$$

Se $2K > m\omega^2$ STABILE

$2K < m\omega^2$ INSTABILE

$2K = m\omega^2$ NON HO EQUILIBRI

Esercizio 3

$$P : \left(x, \; y, \; \sin(x) \right)$$

$$\dot{P} : \left(\dot{x}, \; \dot{y}, \; \cos(x)\dot{x} \right)$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \cos^2(x)\dot{x}^2 \right)$$

$$V = m y \sin(x)$$

$$L = \frac{1}{2} m \left(1 + \cos^2(x) \right) \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - m y \sin(x)$$

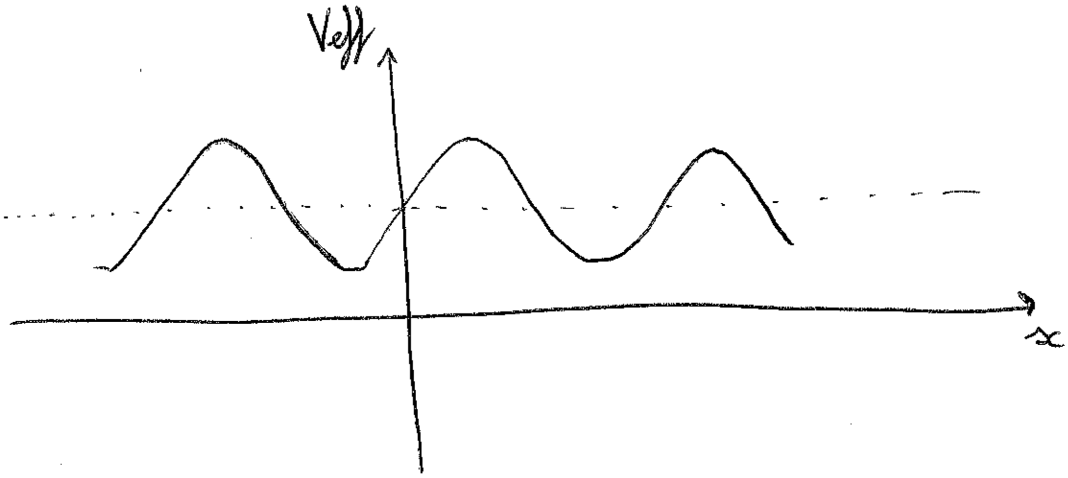
y variabile ciclica $\Rightarrow p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$ COSTANTE DEL MOTO

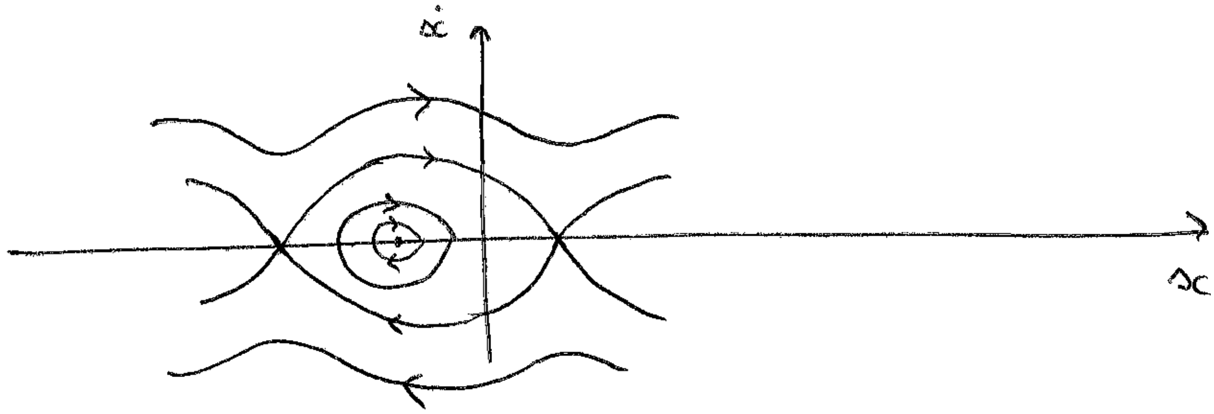
Naturalmente anche $E = T + V$ e' costante del moto

p_y : conservazione della componente lungo asse y della quantita' di moto.

Sostituisco p_y in E.

$$E = \frac{1}{2} m \left(1+\cos^2(x)\right) \dot{x}^2 + \underbrace{\frac{p_y^2}{2m} + m y \sin(x)}_{V_{eff}}$$





Nel sistema completo

$$(x, y, \dot{x}, \dot{y}) = \left(\tfrac{\pi}{2}, y, 0, 0\right) \quad \text{Equilibrio instabile}$$

$$\left(-\tfrac{\pi}{2}, y, 0, 0\right) \quad \text{Equilibrio instabile}$$

$$\left.\begin{matrix} (x, \dot{x}) \in \text{orbite periodiche sistema ridotto} \\ \\ (y, 0) \end{matrix}\right\} \text{periodiche sistema completo.}$$