Improved convergence estimates for the Schröder-Siegel problem

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The Schröder-Siegel problem

Consider an analytic map of $\mathbb{C}^n$ into itself that leaves the origin fixed,

$$x' = \mathbf{\Lambda} x + v_1(x) + v_2(x) + \ldots \ , \quad x \in \mathbb{C}^n \ , \tag{1}$$

- $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with **non-resonant eigenvalues**;
- $v_s(x)$ is a homogeneous polynomial of degree $s + 1$;
- the series is convergent in a neighborhood of the origin of $\mathbb{C}^n$.

Find an analytic near the identity coordinate transformation

$$y_j = x_j + \phi_{1,j}(x) + \phi_{2,j}(x) + \ldots \ , \quad j = 1, \ldots, n \ ,$$

with $\phi_{s,j}$ of degree $s + 1$ which **conjugates the map to its linear part**,

$$y' = \mathbf{\Lambda} y \ .$$

Non-resonance condition

Write the eigenvalues of $\mathbf{\Lambda}$ in exponential form as

$$\lambda_j = e^{\mu_j + i\omega_j} \ , \quad \text{with } \mu_j, \omega_j \in \mathbb{R} \ .$$

Define the sequence $\{\beta_r\}_{r \geq 0}$ of positive real numbers as

$$\beta_0 = 1 \ , \quad \beta_r = \min_{|k|=r+1} \left| e^{\langle k, \mu + i\omega \rangle - \mu_j - i\omega_j} - 1 \right| , \ r \geq 1$$

The eigenvalues are said to be **non-resonant** if $\beta_r \neq 0$ for $r \geq 1$.

This is enough to solve the Schröder problem **in formal sense**.

Known results and motivation of the work

- 1871: formal solution by Schröder in the case $n = 1$, when the Siegel domain is the unit circle, i.e., $\lambda = e^{i\omega}$;
- 1942: convergence proof by Siegel, assuming a strong non-resonance condition of **Diophantine type** (via a delicate number-theoretical lemma, often called Siegel lemma).

In the last decades **two main problems** have been raised:

- optimal **non-resonance condition**;
- size of the **convergence radius** of the transformation to normal form.

The results *via* Yoccoz geometric renormalization

In the case $\mathbf{n=1}$ both questions have been answered exploiting the *geometric renormalization* approach introduced by Yoccoz.

- non-resonant condition: **Bruno's** condition;
- convergence radius:
$$\ln \rho \geq -C\text{Б} + C' \ ,$$
where C' is a constant **independent** of λ.

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The *geometric renormalization methods* can **not** be extended to the case $n > 1$, **for which only the value $\mathbf{C = 2}$ has been found**.

Optimal non-resonant condition

Define the sequence $\{\alpha_r\}_{r\geq 0}$ as

$$\alpha_r = \min_{0\leq s\leq r} \beta_s \ , \quad r \geq 0 \ .$$

Bruno's condition

$$-\sum_{k>0} \frac{\ln \alpha_{2^k-1}}{2^k} = \text{Б} < \infty \ .$$

The Bruno condition **weaker** than the Diophantine one.

Statement of the result

Condition τ

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Condition $\boldsymbol{\tau}$

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Main Theorem

Consider the map (1) and assume that condition $\boldsymbol{\tau}$ holds true. Then there exists a transformation $y = x + \psi(x)$, with ψ analytic at least in the polydisk of radius $\mathbf{B^{-1}e^{-\Gamma}}$, where $B > 0$ is a universal constant, which transforms the map into the **normal form** $y' = \boldsymbol{\Lambda} y$.

Condition $\boldsymbol{\tau}$ *vs.* Bruno's condition

We prove that

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holds true with $\mathbf{C = 1}$ in *any finite dimension*.

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Condition $\boldsymbol{\tau}$ is equivalent to Bruno's one, indeed

$$\Gamma \leq \text{Б} \leq 2\Gamma \ .$$

Condition $\boldsymbol{\tau}$ comes out naturally from our analysis of the **accumulation of small divisors** and turns out to be the **key** that allows us to find the estimate with $\mathbf{C = 1}$.

About the method

This approach extends to the case of **maps** a previous result for the case of the **Poincaré-Siegel** *center problem* (linearization of an analytic system of differential equations in the neighborhood of a fixed point).

We stress that the interest of the method is not limited to the cases mentioned here:

- applications to **KAM theory**: Kolmogorov normal form, elliptic lower dimensional tori in planetary systems;
- proof of **Lyapounov's theorem** on the existence of periodic orbits in the neighborhood of the equilibrium;
- **control** of a particle accelerator model.

Representation of maps

Let $\mathbf{\Lambda} = e^{\mathbf{A}}$ with

$$\mathbf{A} = \mathrm{diag}(\mu_1 + i\omega_1, \ldots, \mu_n + i\omega_n) \quad \text{with } \lambda_j = e^{\mu_j + i\omega_j} \ .$$

We express the linear part of the map as a **Lie series** by introducing the exponential operator $\mathbf{R} = \exp\bigl(L_{\mathbf{A}x}\bigr)$.

Lemma

There exist generating sequences of vector fields $V = \bigl\{V_s(x)\bigr\}_{s\geq 1}$ and $W = \bigl\{W_s(x)\bigr\}_{s\geq 1}$ with $W_s = \mathbf{R}V_s$ such that (1) is represented as

$$x' = \mathbf{R} \circ T_V x \quad \text{and} \quad x' = T_W \circ \mathbf{R}\, x$$

Composition of Lie transforms

Lemma

Let X, Y be generating sequences. Then one has $T_X \circ T_Y = T_Z$ where Z is the generating sequence recursively defined as

$$Z_1 = X_1 + Y_1 \ , \quad Z_s = X_s + Y_s + \sum_{j=1}^{s-1} \frac{j}{s} E^X_{s-j} Y_j \ .$$

Similar to the well known **Baker-Campbell-Hausdorff** formula.

The result is **expressed as a Lie transform** instead of an exponential, which makes the formula *more effective* for our purposes.

Normal form

Consider two maps

$$x' = T_W \circ \mathbf{R}\, x \ , \quad y' = T_Z \circ \mathbf{R}\, y \tag{2}$$

Lemma

Let the generating sequences of the maps (2) coincide up to order $r-1$ and let X_r be a vector field of order r generating the near the identity transformation $y = \exp(L_{X_r} x)$. Then the maps are conjugated up to order r if

$$T_Z = \exp(L_{X_r}) \circ T_W \circ \exp(L_{-\mathbf{R}X_r}) \ .$$

The vector field X_r must satisfy the equation

$$\mathbf{D}X_r = W_r - Z_r \ , \quad \mathbf{D} = \mathbf{R} - \mathbf{1} \ .$$

Schröder-Siegel problem *via* normal form

$$x' = T_{W^{(0)}} \circ \mathbf{R} x \qquad \Longrightarrow \qquad y' = \mathbf{R} y \ .$$

We look for a generating sequence $\{X_r\}_{r\geq 1}$ of vector fields and a corresponding sequence $\{W^{(r)}\}_{r\geq 1}$ satisfying $\mathbf{W_1^{(r)} = \ldots = W_r^{(r)} = 0}$.

The vector field X_r is determined as

$$X_r = \sum_{j=1}^{n} \mathbf{e}_j \sum_{k} \frac{w_{j,k}}{e^{\langle k, \mu + i\omega\rangle - \mu_j - i\omega_j} - 1} x^k \ ,$$

where $w_{j,k}$ are the coefficients of $W_r^{(r-1)}$.

Accumulation of small divisors

Solving the equation for X_r introduces a divisor $\boldsymbol{\alpha}_{\mathbf{r}}$.

We introduce a further sequence $\{\sigma_r\}_{r\geq 0}$ defined as

$$\sigma_0 = 1\ , \quad \sigma_r = \frac{\alpha_r}{r^2}\ ,\ r \geq 1\ .$$

The quantities σ_r are the actual **small divisors** that we must deal with.

$$\sigma_{j_1} \cdots \sigma_{j_q} \qquad \Longleftrightarrow \qquad j_1, \ldots, j_q\ .$$

The problem is to identify the **worst product** among them. The **key** of our argument is to focus our attention on the **indexes** rather than on the actual values of the divisors.

Selection rule

We call $I = \{j_1, \ldots, j_s\}$ a *list of indexes*.

Partial ordering: $I \lhd I'$ in case there is a permutation of the indexes such that the relation $j_m \le j'_m$ holds true for $m = 1, \ldots, s$.

$$I_s^* = \left(\left\lfloor \frac{s}{s} \right\rfloor, \left\lfloor \frac{s}{s-1} \right\rfloor, \ldots, \left\lfloor \frac{s}{2} \right\rfloor \right) .$$

The **allowed combinations of small divisors** are described by

$$\mathcal{J}_{r,s} = \left\{ I = \{j_1, \ldots, j_{s-1}\} \ : \ j_m \in \{0, \ldots, \min(r, s/2)\} \, , \ I \lhd I_s^* \right\} .$$

The condition $I \lhd I_s^*$ is the *selection rule* **S**.

Accumulation of small divisors

We associate to the sets of indexes $\mathcal{J}_{r,s}$ the sequence of positive real numbers $T_{r,s}$ defined as

$$T_{0,s} = 1 \ , \quad T_{r,s} = \max_{I \in \mathcal{J}_{r,s}} \prod_{j \in I} \frac{1}{\sigma_j} \ , \quad 0 < r \le s \ .$$

Lemma

Let λ satisfy condition $\boldsymbol{\tau}$. Then the sequence $T_{r,s}$ is bounded by

$$T_{r,s} \le \gamma^s e^{s\Gamma} \ , \quad \frac{1}{\sigma_s} T_{r,s} \le \gamma^s e^{s\Gamma}$$

with some positive constant γ not depending on λ.

Bound on the norms of the generating functions

Iterative Lemma

Assume that the sequence $W^{(0)}$ of vector fields satisfies $\|W_s^{(0)}\| \le \frac{C_0^{s-1}A}{s}$ with some constants $A > 0$ and $C_0 \ge 0$. Then the sequence of vector fields $\{X_r\}_{r\ge1}$ that for every r give the normal form $W^{(r)}$ satisfies the following estimates: there exists a bounded monotonically non decreasing sequence $\{C_r\}_{r\ge1}$ of positive constants, with $C_r \to C_\infty < \infty$ for $r \to \infty$, such that we have

$$\|X_r\| \le T_{r-1,r}\frac{C_{r-1}^{r-1}A}{r\alpha_r}\ , \quad \|W_s^{(r)}\| \le T_{r,s}\frac{C_r^{s-1}A}{s}\ .$$

The sequence may be recursively defined as

$$C_1 = 2C_0 + 16A\ , \quad C_r = \left(1+\frac{1}{r^2}\right)^{1/r}\left(1+\frac{1}{r}\right)^{1/r}C_{r-1}\ ,$$

so that one has $C_r > 16A$.

Conclusion

Result

We prove that, assuming the non-resonant condition $\boldsymbol{\tau}$

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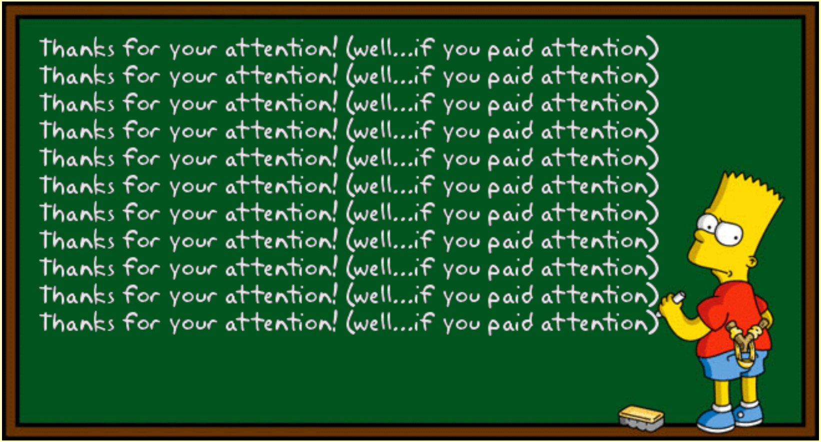
Remarks

Condition $\boldsymbol{\tau}$ is equivalent to the Bruno's one, that has been proved to be **optimal** in the case $n = 1$.

We obtain $\mathbf{C = 1}$ in any finite dimension and we recall that in the case $n = 1$ the **optimal value** has been proved to be $C = 1$.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