Explicit Construction of Elliptic Tori for Planetary Systems

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Based on a research work in collaboration with
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The KAM Theory

A.N. Kolmogorov: *Preservation of conditionally periodic movements with small change in the Hamilton function*, D.A.N. SSSR, 98, 527 (1954).

The theorem apply to quasi-integrable *non-degenerate* Hamiltonian system and claims that **if the perturbation is small enough then there exists a set of invariant tori of large measure**.

The relevance of Kolmogorov's result for the planetary problem has been soon emphasized by Arnold and Moser. Arnold produced out a proof taking into account the *degeneration* of the unperturbed Hamiltonian which occurs in the *planetary case*.

These facts marked the beginning of the so called **KAM theory**.

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In recent years the estimates for the applicability of Kolmogorov's theorem to realistic models of some part of the Solar System have been improved by some authors.

Celletti, Chierchia: *KAM stability and Celestial Mechanics*, AMS, (2007).

Locatelli, Giorgilli: *Invariant tori in the Sun-Jupiter-Saturn system*, DCDS-B, (2007).

Invariant Tori of maximal dimension

Kolmogorov's theorem applies to quasi-integrable *non-degenerate* Hamiltonian systems.

In the planetary problem, due to the degeneration of Kepler's motions, we have only n mean motion frequencies.

Arnold's planetary KAM theorem, using essentially a non linear reformulation of Lagrange's and Laplace's theory on secular motions of the nodes and perihelia, reintroduces the $2n$ frequencies, that are missing in the Kepler's approximation.

Since the birth of the KAM theory, **invariant tori** are expected to be the fundamental dynamical objects which explain the quasi-periodicity of the planetary motions of our Solar System.

Low dimensional Invariant Tori

Among the consequences of the statements concerning the invariant tori of maximal dimension, it may be expected that in the limit case of small circular orbits also n-dimensional invariant tori should exists.

The proof of such a statement requires new ideas and analytical tools.

The case of the planetary system has been recently treated by Biasco, Chierchia and Valdinoci for the spatial three-body planetary problem and for a planar system with a central star and n planets.

Their *deep theoretical approach* is related to a theorem due to Pöschel assuring the existence of low dimensional elliptic tori.

However, their method is *not suitable for explicit applications*, even if one is interested just in finding the location of the *elliptic invariant tori*.

The Kolmogorov's Scheme (constructive approach)

The original scheme introduced by Kolmogorov is in a much better position for what concerns the translation in an **explicit algorithm** for the construction of invariant tori.

We adapt the Kolmogorov's algorithm, in order to construct a suitable normal form related to the elliptic tori. Moreover, this will allow us to explicitly integrate the motions on those invariant surfaces, by using a so called semi-analytic procedure.

Let us emphasize that here we focus just on a direct application to a planetary system. Thus, we will check the effectiveness of our semi-analytic procedure, by calculating a finite number of steps of the algorithm by algebraic manipulations on a calculator.

The planar SJSU system

In order to apply our methods to a model similar to our outer Solar System, we study an approximation of the planar Sun-Jupiter-Saturn-Uranus system.

Let us consider four point bodies with masses m_0, m_1, m_2, m_3, mutually interacting according to Newton's gravitational law.

The indexes 0, 1, 2, 3 correspond to Sun, Jupiter, Saturn and Uranus, respectively.

The Hamiltonian of the planetary system

The Hamiltonian is

$$F(\mathbf{r},\tilde{\mathbf{r}}) = T^{(0)}(\tilde{\mathbf{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\tilde{\mathbf{r}}) + U^{(1)}(\mathbf{r})\,,$$

where $\mathbf{r}$ are the heliocentric coordinates and $\tilde{\mathbf{r}}$ the conjugated momenta.

$$T^{(0)}(\tilde{\mathbf{r}}) = \frac{1}{2}\sum_{j=1}^{3}\|\tilde{\mathbf{r}}_j\|^2\left(\frac{1}{m_0}+\frac{1}{m_j}\right)\,,$$

$$U^{(0)}(\mathbf{r}) = -\mathcal{G}\sum_{j=1}^{3}\frac{m_0 m_j}{\|\mathbf{r}_j\|}\,,$$

$$T^{(1)}(\tilde{\mathbf{r}}) = \frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_2}{m_0}+\frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_3}{m_0}+\frac{\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_3}{m_0}\,,$$

$$U^{(1)}(\mathbf{r}) = -\mathcal{G}\left(\frac{m_1 m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|}+\frac{m_1 m_3}{\|\mathbf{r}_1-\mathbf{r}_3\|}+\frac{m_2 m_3}{\|\mathbf{r}_2-\mathbf{r}_3\|}\right)\,.$$

The Poincaré variables in the plane

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j) \qquad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j)$$

where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and perihelion argument of the j-th planet, respectively.

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$$\underbrace{\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j}_{\text{fast variables}}$$

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$$F = F_0 + F_1 = F_0 + U^{(1)} + T^{(1)}$$

$$F_0 = -\sum_{i=1}^{n} \frac{\mu_i^2 \beta_i^3}{2\Lambda_i^2} \qquad \text{integrable part}\,,$$

$$U^{(1)} = -G \sum_{0<i<j} \frac{m_i m_j}{\Delta_{ij}} \qquad \text{perturbation (main term)}\,,$$

$$T^{(1)} = \sum_{0<i<j} \frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0} \qquad \text{perturbation (complementary term)}\,.$$

We need to expand all this terms in power series!

How to expand the Hamiltonian

1. We proceed by following the approach described in L.&G., *CeMDA*, 2000 and L.&G., *DCDS-B*, 2007.
2. Choose a Λ^* such that

$$\left.\frac{\partial\langle F\rangle_{\lambda}}{\partial\Lambda_j}\right|_{\substack{\Lambda=\Lambda^* \\ \xi=\eta=0}} = \omega_j\,, \qquad j=1,2,3\,.$$

   - $\langle.\rangle_{\lambda}$ means the average over the fast angles,
   - ω_j are the fundamental frequencies of the mean motion.
3. Introduce new actions $L_j = \Lambda_j - \Lambda_j^*$.
4. Perform the canonical transformation $\mathcal{T}_F$ translating the fast actions.
5. Expand the Hamiltonian in power series of $\mathbf{L}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and in Fourier series of $\boldsymbol{\lambda}$.

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The expansion of the Hamiltonian

For what concerns the classical expansions of the Hamiltonian in canonical variables we essentially follow

- Laskar & Robutel, *CeMDA*, 1995,
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$$H^{(\mathcal{T}_F)} = \boldsymbol{\omega} \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where $h_{j_1,0}^{(Kep)}$ is a hom. pol. of degree j_1 in $\mathbf{L}$ and

$$h_{j_1,j_2}^{(\mathcal{T}_F)} \text{is a} \begin{cases} \text{hom. pol. of degree } j_1 \text{ in } \mathbf{L}\,, \\ \text{hom. pol. of degree } j_2 \text{ in } \boldsymbol{\xi}, \boldsymbol{\eta}\,, \\ \text{with coeff. that are trig. pol. in } \boldsymbol{\lambda}\,. \end{cases}$$

Truncation limits of the expansion

Using a slightly different notation, the Hamiltonian is the following,

$$H = \boldsymbol{\omega} \cdot \mathbf{L} + \sum_{j_1 \geq 2} h_{j_1,0}(\mathbf{L}) + \mu \sum_{j_1 \geq 0} \sum_{j_2 \geq 0} f_{j_1,j_2}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

and we truncate the expansions adopting the following criteria:

- the Keplerian part is expanded up to the quartic terms;
- the terms f_{j_1,j_2} include:
  - the terms having degree j_1 in the actions $\mathbf{L}$ with $j_1 \leq 3$;
  - all terms having degree j_2 in the secular variables $(\boldsymbol{\xi}, \boldsymbol{\eta})$, with j_2 such that $2j_1 + j_2 \leq 8$;
  - all terms up to the trigonometric degree 18 with respect to the angles $\boldsymbol{\lambda}$.

The equation of motion

Consider any point of the type $(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0}) \in (\mathbf{0}, \mathbb{T}^3, \mathbf{0}, \mathbf{0})$, we have

$$\dot{L}_j = -\frac{\partial f_{0,0}}{\partial \lambda_j}(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0}) \qquad \dot{\xi}_j = -\frac{\partial f_{0,1}}{\partial \eta_j}(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0})$$

$$\dot{\lambda}_j = \omega_j + \frac{\partial f_{1,0}}{\partial L_j}(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0}) \qquad \dot{\eta}_j = \frac{\partial f_{0,1}}{\partial \xi_j}(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0})$$

so, in order to make the manifold $(\mathbf{0}, \mathbb{T}^3, \mathbf{0}, \mathbf{0})$ invariant, we need to **kill the terms** $f_{0,0}$, $f_{0,1}$ and $f_{1,0}$.

Rearranging the Hamiltonian

Our aim is to remove the unwanted terms via normal form, using the composition of Lie series to do the canonical transformation.

To apply our algorithm we have to rearrange the $f_{0,2}$ term of the expansion, isolating the “diagonal component”, as

$$f_{0,2} = \boldsymbol{\Omega} \cdot \zeta + \hat{f}_{0,2}$$

where $\zeta_j = \left(\xi_j^2 + \eta_j^2\right)/2$.

We also split the perturbation terms by trigonometric degree,

$$f_{j_1,j_2} = \sum_{r>0} f_{j_1,j_2}^r$$

where f_{j_1,j_2}^r contains only terms with trigonometric degree, with respect to the fast angles $\boldsymbol{\lambda}$, not greater than $r\,K$, with K a fixed parameter.

The Hamiltonian at order $r-1$

The Hamiltonian at order $r-1$, say $H^{(r-1)}$, can be written as

$$H^{(r-1)} = \boldsymbol{\omega}\cdot\mathbf{L} + \boldsymbol{\Omega}^{(\mathbf{r-1})}\cdot\zeta + \sum_{j_1\geq 2} h_{j_1,0}(\mathbf{L})$$
$$+ \sum_{s\geq r} f_{0,0}^{(r-1,s)}(\boldsymbol{\lambda}) + \sum_{s\geq r} f_{0,1}^{(r-1,s)}(\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) + \sum_{s\geq r} f_{0,2}^{(r-1,s)}(\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})$$
$$+ \sum_{s\geq r} f_{1,0}^{(r-1,s)}(\mathbf{L},\boldsymbol{\lambda}) + \sum_{j_1\geq 0}\sum_{j_2\geq 0}\sum_{s>0} f_{j_1,j_2}^{(r-1,s)}(\mathbf{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta})$$

where the first upper index clearly refers to the normalization order.

We now describe the r normalization step.

The Kolmogorov like procedure

- First step

$$\mathcal{L}_{X_0^r}\left(\boldsymbol{\omega}\cdot\mathbf{L}\right)+f_{0,0}^{(r-1,r)}=0$$

$$\mathcal{L}_{\xi^r\cdot\boldsymbol{\lambda}}\langle f_{2,0}^{(r-1,r)}\rangle_{\boldsymbol{\lambda}}+\langle f_{1,0}^{(r-1,r)}\rangle_{\boldsymbol{\lambda}}=0$$

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- Second step

$$\mathcal{L}_{X_1^r}\left(\boldsymbol{\omega}\cdot\mathbf{L}+\boldsymbol{\Omega}^{(\mathbf{r-1})}\cdot\boldsymbol{\zeta}\right)+f_{0,1}^{(r-1,r)}=0$$

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- Second step

$$\mathcal{L}_{X_1^r}\left(\boldsymbol{\omega}\cdot\mathbf{L}+\boldsymbol{\Omega}^{(\mathbf{r-1})}\cdot\boldsymbol{\zeta}\right)+f_{0,1}^{(r-1,r)}=0$$

- Third step

$$\mathcal{L}_{X_2^r}\left(\boldsymbol{\omega}\cdot\mathbf{L}+\boldsymbol{\Omega}^{(\mathbf{r-1})}\cdot\boldsymbol{\zeta}\right)+f_{0,2}^{(r-1,r)}-\langle f_{0,2}^{(r-1,r)}\rangle_{\boldsymbol{\lambda}}=0$$

$$\mathcal{L}_{Y_2^r}\left(\boldsymbol{\omega}\cdot\mathbf{L}\right)+f_{1,0}^{(r-1,r)}=0$$

$$\mathcal{L}_{D^r}\left(\boldsymbol{\Omega}^{(\mathbf{r-1})}\cdot\boldsymbol{\zeta}+\langle f_{0,2}^{(r-1,r)}\rangle_{\boldsymbol{\lambda}}^{D}\right)+\langle f_{0,2}^{(r-1,r)}\rangle_{\boldsymbol{\lambda}}^{ND}=0$$

The Small Divisors & the Melnikov conditions

- First step

$$\widetilde{\alpha}_r = \min_{0<|\mathbf{k}|\leq rK} \left(|\mathbf{k}\cdot\boldsymbol{\omega}|\right)$$

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- Second step (*first Melnikov's condition*)

$$\hat{\alpha}_r = \min_{\substack{|\mathbf{k}|\le rK \\ |\boldsymbol{\ell}|=1}} \left(|\mathbf{k}\cdot\boldsymbol{\omega} + \boldsymbol{\ell}\cdot\boldsymbol{\Omega}| \right)$$

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- Third step (*second Melnikov's condition*)

$$\bar{\alpha}_r = \min_{\substack{|\mathbf{k}|\leq rK \\ |\boldsymbol{\ell}|=2}} \left(|\mathbf{k}\cdot\boldsymbol{\omega} + \boldsymbol{\ell}\cdot\boldsymbol{\Omega}|\right)$$

The Normalization algorithm

Defining the three generating functions as

$$\chi_0^r = X_0^r + \xi^r \cdot \boldsymbol{\lambda}\,,$$
$$\chi_1^r = X_1^r\,,$$
$$\chi_2^r = X_2^r + Y_2^r + D^r\,,$$

we can define the operator $\mathcal{C}^r$ as

$$\mathcal{C}^r = \exp\left(\mathcal{L}_{\chi_2^r}\right)\exp\left(\mathcal{L}_{\chi_1^r}\right)\exp\left(\mathcal{L}_{\chi_0^r}\right)\ldots\exp\left(\mathcal{L}_{\chi_2^1}\right)\exp\left(\mathcal{L}_{\chi_1^1}\right)\exp\left(\mathcal{L}_{\chi_0^1}\right)\,,$$

where the generating functions χ_0^r, χ_1^r and χ_2^r have trigonometric degree smaller than r and are of degree j_1 in the actions $\mathbf{L}$ and j_2 in the secular variables $(\boldsymbol{\xi},\boldsymbol{\eta})$, with $2j_1 + j_2 = 0,\ 1,\ 2$, respectively.

The Elliptic Invariant Torus

Let us denote by $\mathcal{E}$ the function

$$\mathcal{E}(L, \lambda, \xi, \eta) = (x, y, p_x, p_y)\,,$$

that transform the point (L, λ, ξ, η) into the corresponding position-momenta coordinates.

The sets of initial conditions corresponding to the elliptic torus is given by $\mathcal{E} \circ \mathcal{C}^{\infty}(0, \lambda, 0, 0)$, with $\lambda \in \mathbf{T}^3$.

The Fourier spectrum of the motions on elliptic tori is strongly characteristic, in fact **just the fast frequencies and their linear combinations can show up**. This simple remark allows us to check the accuracy of our results by using frequency analysis.

The Approximated Torus

With our procedure, we are able to produce several approximations of the elliptic torus. Let us consider a generic point on the torus, e.g. $(0,0,0,0)$.

Starting with the trivial approximation

$$\mathcal{E} \circ \mathcal{C}^0(0,0,0,0)\,,$$

by using computer algebra we have been able to construct the approximation of the elliptic torus corresponding to

$$\mathcal{E} \circ \mathcal{C}^9(0,0,0,0)\,.$$

The Frequency Analysis

Starting from the integration of the initial conditions

$$\mathcal{E} \circ \mathcal{C}^0(0,0,0,0) \qquad \text{and} \qquad \mathcal{E} \circ \mathcal{C}^9(0,0,0,0)\,,$$

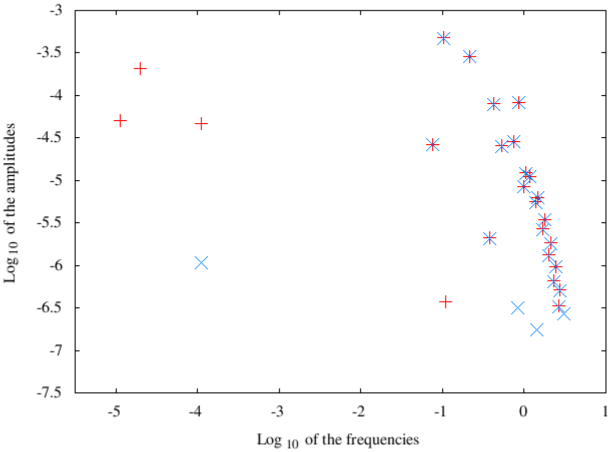
we perform the frequency analysis of the three signals

$$\big(\xi_j(t), \eta_j(t)\big) \qquad \text{for } j = 1\,,\ 2\,,\ 3\,,$$

for the two different starting points.

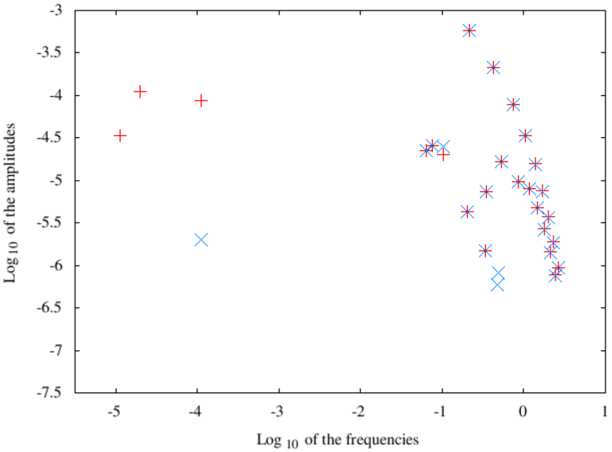
We report our results in the following graphs.

Jupiter



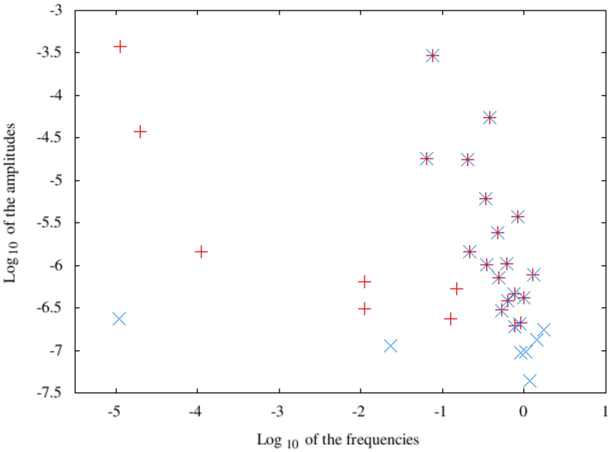
The red symbols refer to the trivial approximation while the blue ones refer to the approximated elliptic torus.

Saturn



The red symbols refer to the trivial approximation while the blue ones refer to the approximated elliptic torus.

Uranus



The red symbols refer to the trivial approximation while the blue ones refer to the approximated elliptic torus.

Conclusions

The comparison of the frequency analysis of the trivial approximation and of the approximated elliptic torus show the *decay of the secular frequencies* after the normalization procedure.

In the approximated elliptic torus, we can roughly say that secular frequencies have no values greater than 10^{-3}.

We also remark that *nearly all the components* correspond to *linear combinations* of fast frequencies.

Each plot highlights the occurrence of *just one component* corresponding to a secular frequency, with an amplitude that is *much smaller* than the main ones.

This shows that our algorithm is *very effective*. However, let us recall that the occurrence of secular components are unavoidable in a practical application, due to truncations and numerical errors.

Thanks for your attention!