Towards stability results for planetary problems with more than three bodies

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Based on a research work in collaboration with Antonio Giorgilli [b]

Chaos in the Sun–Jupiter–Saturn–Uranus–Neptune system

By numerical integrations and calculation of Lyapunov time, Sussman & Wisdom (*Science*, 1992) showed that the giant planets subsystem (hereafter SJSUN system) is **chaotic**, but small changes of the initial conditions can yield quasi–periodic motions.

Murray & Holman (*Science*, 1999) pointed out the *overlap of some three–body resonances* as the responsible of this chaotic phenomenon. For instance, the actual value of the Uranus semi-major axis is located very close to the center of the cluster of the following type of resonances:

$$3n_J - 5n_S - 7n_U + [(3-j)g_J + 6g_S + jg_U] \qquad \text{with} \;\; j = 0,\,1,\,2,\,3\;,$$

where n stands for the mean motion frequency of a planet and g means the (secular) frequency of its perihelion argument.
The same applies also to other cluster of resonances "generated" by the $(3, -5, -7)$ mean motion resonance; some of them include also the frequencies related to the longitude of the nodes.

Dynamics of the SJSUN and SJSU systems

Remark:

The triple $(3, -5, -7)$ mean motion resonance is so relevant, because Saturn is close to the celebrated $5:2$ resonance with Jupiter, while Uranus is near to the $7:1$. Moreover, $2n_J - 5n_S \simeq 7n_U - n_J$.

Looking at the size of the resonant terms, M.&H. roughly evaluated the time T_{ej} needed by Uranus to be ejected as $T_{ej} \sim 10^{18}$.
M.&H. studied the behavior of many *fictitious* Jovian planets system, by changing a little the initial value of the Uranus semi-major axis a_U.

Result (given by numerical explorations):

When a_U ranges between 19.18 and 19.35 AU, some regions looks filled by quasi–periodic motions (i.e. with Lyap. time $> 10^8$), while some other regions are chaotic due to the effect of clusters of three–body resonances.

This result *(qualitatively) still holds true* either *in the planar case* or *after having removed Neptune*.
On the other hand, in that whole region M.&H. did not detect chaotic motions *in the planar case without Neptune*.

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3. The form of the secular Hamiltonian is now the following:

$$H(\xi_1, \xi_2, \eta_1, \eta_2) = \mathcal{P}_2(\xi_1, \xi_2, \eta_1, \eta_2) + \mathcal{P}_4(\xi_1, \xi_2, \eta_1, \eta_2) + \mathcal{P}_6 \ldots \ ,$$

where $(\xi, \eta) = \mathcal{O}(ecc.)$ and $\mathcal{P}_{2j}$ is a homogeneous polynomial of degree $2j$ in the arguments. Therefore, it is useful to first "diagonalize" the quadratic part by a linear canonical transformation.

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5. In action–angle coordinates such that $(\xi, \eta) = \sqrt{2I}(\cos\varphi, \sin\varphi)$:

$$H(I_1, I_2, \varphi_1, \varphi_2) = \nu_1 I_1 + \nu_2 I_2 + \mathcal{P}_4(I_1, I_2) + \mathcal{P}_6(I_1, I_2) + \mathcal{P}_8(I_1, I_2, \varphi_1, \varphi_2) + \ldots \ ,$$

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6. "Shift the actions origin" so that the integrable part is centered about the torus corresponding to the wanted secular frequencies.
7. Perform the usual "Kolmogorov's normalization algorithm".

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The previous scheme has been successfully applied to the *secular part up to order* 2 *in the masses* of the Sun–Jupiter–Saturn system (L.&G., *Cel. Mech. Dyn. Astr.*, 2000). This allowed us to *rigorously* prove the existence of a couple of invariant tori bounding the motion, so to ensure the stability of the secular system.

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Question:

What is the secular part *up to order* 2 *in the masses* of a planetary Hamiltonian?

A.: See the next two slides!

Construction of KAM tori for a complete 3–body system

Let us assume the following hypotheses about our Hamiltonian:

- the correspondence between actions and frequencies are **non–degenerate** both for the Keplerian and the secular parts;
- the frequencies are **non–resonant** enough;
- the ratio of the biggest planet mass over the star mass and both the eccentricities and the inclinations are **small enough**.

Therefore, the same preliminary steps 1–6 (except the average) previously described are suitable to successfully start the "Kolmogorov's normalization algorithm" on the **complete** Hamiltonian.

This allowed us to perform a *"semi–analytic" construction* of the KAM tori for a *concrete three–body planetary problem* and, then, we calculated its quasi–periodic motion (L.&G., *Reg. Chaot. Dyn.*, 2005).

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Question:

What is a *"semi–analytic" procedure*?
A.: It is not a proof! It is the description of an algorithm allowing us to explicitly calculate analytic expansions of the solution by computer algebra.

Construction of KAM tori for the complete SJS system

The previously described approach *cannot be straightforwardly applied to the* **real Sun–Jupiter–Saturn system**. To deal with this real system, it is mandatory to perform *a preliminary reduction of the perturbing terms depending on the mean motion angles*, **before** starting the manipulation of the secular coordinates (P. Robutel, *Ph.D. thesis*, 1993).

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We did it in a "Kolmogorov's like" way, i.e. we removed the main terms which are *actually depending on the mean motion angles, at most linear in the actions* (L_1, L_2) and up to a fixed degree in eccentricities, by using a couple of canonical transformations.

After those transformations, the average of the Hamiltonian can provide an approximation of the secular motions, that is valid *up to order* 2 *in the masses*. Moreover, we can apply the same algorithm for the construction of KAM tori, as described in the previous slide.

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Result (given by our semi-analytic method):

The procedure constructing the KAM torus is shown to be convergent and the corresponding quasi–periodic flow provide a *good approximation of the motion* of the **real SJS system** (L.&G., *DCDS–B*, 2007).

Normal forms approach to the SJSU and SJSUN systems

Question:

What did we learn from the study of the Sun–Jupiter–Saturn system?

A.1: A careful handling of *the secular part of the Hamiltonian is crucial*.

A.2: Before starting to manipulate the secular part of the Hamiltonian, we need to perform a preliminary transformation “à la Kolmogorov” to reduce the main part of the perturbation depending on the fast angles.

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Strategy options (after having added, at least, Uranus):

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2. in the neighborhood of an elliptic torus;
3. nearby a KAM torus (as in G.,L.&S., *Cel. Mech. Dyn. Astr.*, 2009).

Application to the planar SJSU system

- The **planar** Sun–Jupiter–Saturn–Uranus (shortly, SJSU) system.
- Huge amount of calculations in the "Kolmogorov's like" step.
- **No** reduction of the angular momentum.
- Expansion of the Hamiltonian using an algebraic manipulator.
- Study of the secular dynamics.
- Estimate of the "stability time".

The Hamiltonian of the planetary system

The Hamiltonian is

$$F(\mathbf{r},\tilde{\mathbf{r}}) = T^{(0)}(\tilde{\mathbf{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\tilde{\mathbf{r}}) + U^{(1)}(\mathbf{r})\,,$$

where $\mathbf{r}$ are the heliocentric coordinates and $\tilde{\mathbf{r}}$ the conjugated momenta.

$$T^{(0)}(\tilde{\mathbf{r}}) = \frac{1}{2}\sum_{j=1}^{3}\|\tilde{\mathbf{r}}_j\|^2\left(\frac{1}{m_0}+\frac{1}{m_j}\right)\,,$$

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The Poincaré variables in the plane

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j} \sqrt{\mathcal{G}(m_0 + m_j) a_j} \qquad \lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2}} \cos(\omega_j) \quad \eta_j = -\sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2}} \sin(\omega_j)$$

where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and perihelion argument of the j–th planet, respectively.

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$$F = F_0 + F_1 = F_0 + U^{(1)} + T^{(1)}$$

$$F_0 = -\sum_{i=1}^{n} \frac{\mu_i^2 \beta_i^3}{2\Lambda_i^2} \qquad \text{integrable part}\,,$$

$$U^{(1)} = -G \sum_{0<i<j} \frac{m_i m_j}{\Delta_{ij}} \qquad \text{perturbation (main term)}\,,$$

$$T^{(1)} = \sum_{0<i<j} \frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0} \qquad \text{perturbation (complementary term)}\,.$$

We need to expand all this terms in power series!

How to expand the Hamiltonian

1. We proceed by following the approach described in L.&G., *CeMDA*, 2000 and L.&G., *DCDS–B*, 2007.
2. Choose a Λ^* such that

$$\left.\frac{\partial \langle F \rangle_{\lambda}}{\partial \Lambda_j}\right|_{\substack{\Lambda=\Lambda^* \\ \xi=\eta=0}} = n_j^* \,, \qquad j = 1, 2, 3 \,.$$

   - $\langle . \rangle_{\lambda}$ means the average over the fast angles ,
   - n_j^* are the fundamental frequencies of the mean motion .
3. Introduce new actions $L_j = \Lambda_j - \Lambda_j^*$.
4. Perform the canonical transformation $\mathcal{T}_F$ translating the fast actions.
5. Expand the Hamiltonian in power series of $\mathbf{L}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and in Fourier series of $\boldsymbol{\lambda}$.

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1. We proceed by following the approach described in L.&G., *CeMDA*, 2000 and L.&G., *DCDS–B*, 2007.
2. Choose a Λ^* such that

$$\frac{\partial \langle F \rangle_{\lambda}}{\partial \Lambda_j}\bigg|_{\substack{\Lambda=\Lambda^* \\ \xi=\eta=0}} = n_j^* \,, \qquad j = 1, 2, 3\,.$$

   - $\langle . \rangle_{\lambda}$ means the average over the fast angles ,
   - n_j^* are the fundamental frequencies of the mean motion .
3. Introduce new actions $L_j = \Lambda_j - \Lambda_j^*$.
4. Perform the canonical transformation $\mathcal{T}_F$ translating the fast actions.
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The expansion of the Hamiltonian

For what concerns the classical expansions of the Hamiltonian in canonical variables we essentially follow

- Laskar & Robutel, *CeMDA*, 1995,
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$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where $h_{j_1,0}^{(Kep)}$ is a hom. pol. of degree j_1 in $\mathbf{L}$ and

$$h_{j_1,j_2}^{(\mathcal{T}_F)} \text{is a} \begin{cases} \text{hom. pol. of degree } j_1 \text{ in } \mathbf{L}\,, \\ \text{hom. pol. of degree } j_2 \text{ in } \boldsymbol{\xi}, \boldsymbol{\eta}\,, \\ \text{with coeff. that are trig. pol. in } \boldsymbol{\lambda}\,. \end{cases}$$

Truncation limits of the expansion

This is the Hamiltonian,

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

Truncation limits of the expansion

This is the computed Hamiltonian,

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{2} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{1} \sum_{j_2=0}^{12} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where we also truncate all the coefficients with harmonics of degree greater than 16.

These are the lowest limits to include the fundamental features of the system.

Partial preliminary reduction of the perturbation

$\lceil \cdot \rceil_{\boldsymbol{\lambda}:K_F}$ means the truncation of the harmonics of degree greater than K_F.

First step

$$\begin{cases} \mathbf{n}^* \cdot \dfrac{\partial \chi_1^{(\mathcal{O}2)}}{\partial \boldsymbol{\lambda}} + \mu \displaystyle\sum_{j_2=0}^{6} \left\lceil h_{0,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \\ \widetilde{H} = \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} H = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}^j_{\chi_1^{(\mathcal{O}2)}} H \,. \end{cases}$$

Second step

$$\begin{cases} \mathbf{n}^* \cdot \dfrac{\partial \chi_2^{(\mathcal{O}2)}}{\partial \boldsymbol{\lambda}} + \mu \displaystyle\sum_{j_2=0}^{6} \left\lceil \tilde{h}_{1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \\ H^{(\mathcal{O}2)} = \exp \mathcal{L}_{\chi_2^{(\mathcal{O}2)}} \circ \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} H \,. \end{cases}$$

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Why these limits?

The secular variables:

$$\widetilde{H} \qquad \chi_2^{(\mathcal{O}2)} \qquad \longrightarrow \qquad H^{(\mathcal{O}2)}$$

The fast angles:

$(3, -5, -7)$ harmonics of order 15

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\downarrow & \downarrow & & & \downarrow \\
\cos(1\lambda_1 - 7\lambda_3) & \sin(1\lambda_1 - 7\lambda_3) & & & \text{secular terms}
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The fast angles:

$$(3, -5, -7) \text{ harmonics of order } 15 < \boxed{16}.$$

The Hamiltonian *up to order two in the masses*

- $H^{(\mathcal{O}2)}$ is the Hamiltonian up to order two in the masses.
- **No** terms corresponding to any *triple resonances* **before** the "Kolmogorov's like" step.
- The "Kolmogorov's like" step introduce the *triple resonances*, in particular the $(3, -5, -7)$ resonance.
- Small limits **don't** mean small expansion!
- After the "Kolmogorov's like" step, we have 94 109 751 coefficients.

The secular part *up to order two in the masses*

- Reduction to the secular system:
  - average over the fast angles $\boldsymbol{\lambda}$, and put $\mathbf{L} = 0$;
  - hereafter, we are considering a system with *three degrees of freedom*.
- From the D'Alembert rules, it follows that

$$H^{(sec)} = H_0 + H_2 + H_4 + \ldots \,,$$

  where H_{2j} is a hom. pol. of degree $(2j+2)$ in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, $\forall\, j \in \mathbf{N}$.
- $\boldsymbol{\xi} = \boldsymbol{\eta} = 0$ is an elliptic equilibrium point.
- We diagonalize the quadratic term by a linear canonical transformation $\mathcal{D}$:

$$H_2^{(\mathcal{D})} = \sum_{j=1}^{3} \frac{\nu_j}{2} \left(\xi_j^2 + \eta_j^2\right) \,.$$

- Hereafter, we simply denote with H the secular Hamiltonian having the quadratic part in diagonal form.

Birkhoff normalization of the secular Hamiltonian

- Consider the secular Hamiltonian having the quadratic part in diagonal form:

$$H = H_0 + H_2 + H_4 + \ldots .$$

- Focus on the actions $\Phi_j = \frac{1}{2}\left(\xi_j^2 + \eta_j^2\right)$ with $j = 1, 2, 3$.

- Perform the Birkhoff normalization up to order N :

$$H^{(N)} = Z_0^{(N)} + Z_2^{(N)} + \ldots + Z_N^{(N)} + R_{N+1}^{(N)} + \ldots \ ,$$

where $Z_0^{(N)}$, $Z_2^{(N)}$, $\ldots$, $Z_N^{(N)}$ just depend on Φ_1 , Φ_2 , Φ_3 .

- The time derivative

$$\dot{\Phi}_j = \{\Phi_j,\, H\} = \sum_{j>N} \left\{\Phi_j,\, R_j^{(N)}\right\} \simeq \left\{\Phi_j,\, R_{N+1}^{(N)}\right\}$$

Study of the stability of the secular Hamiltonian

We have

$$\|\Phi(t)-\Phi(0)\| \le \left| \sup_{(\boldsymbol{\xi},\boldsymbol{\eta})\in\Delta_{\rho\mathbf{R}}} \dot{\Phi}(\boldsymbol{\xi},\boldsymbol{\eta}) \right| \, |t| \ ,$$

where $\Delta_{\rho\mathbf{R}} = \left\{ (\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{R}^6 : \, \xi_j^2+\eta_j^2 \le \rho^2 R_j^2, \ j=1,2,3 \right\}$.

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We want

$$\|\Phi(t) - \Phi(0)\| < \rho - \rho_0 \qquad \forall \ |t| < T \ ,$$

where T is a (eventually long) “stability time”.

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Take a function

$$f(\mathbf{x},\,\mathbf{y}) = \sum_{|\mathbf{j}+\mathbf{k}|=s} f_{\mathbf{j},\mathbf{k}}\,\mathbf{x}^{\mathbf{j}}\mathbf{y}^{\mathbf{k}}$$

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define the norm

$$\|f\|_{\mathbf{R}} = \sum_{|\mathbf{j}+\mathbf{k}|=s} |f_{\mathbf{j},\mathbf{k}}|\mathbf{R}^{\mathbf{j}+\mathbf{k}}$$

The “stability time”

$$T(\rho_0, \rho, N) \lesssim \frac{\rho - \rho_0}{\left\| \left\{ \Phi_j ,\ R_{N+1}^{(N)} \right\} \right\|_{\mathbf{R}} \rho^{N+3}} ,$$

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- Fix ρ_0 and N, a simple analytical estimate allows us to obtain $\rho_{opt}(\rho_0, N)$.

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- Fix ρ_0, by using $\rho_{opt}(\rho_0, N)$ and by numerically studying the "stability time" as a function of the normalization step, we obtain $N_{opt}(\rho_0)$.

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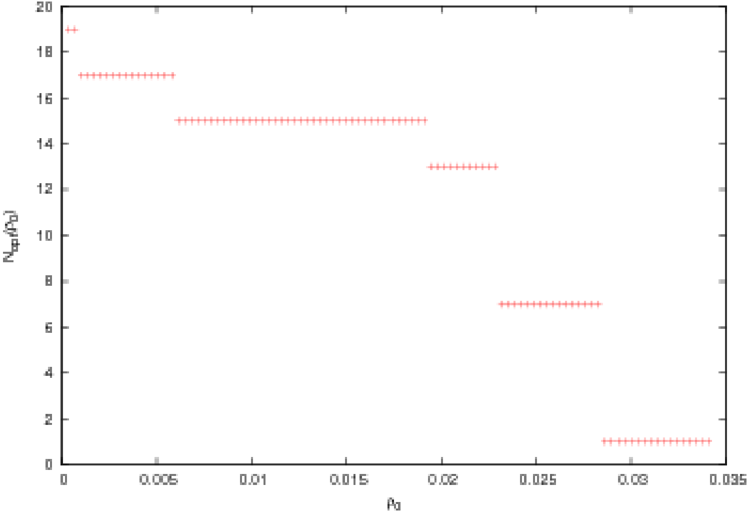
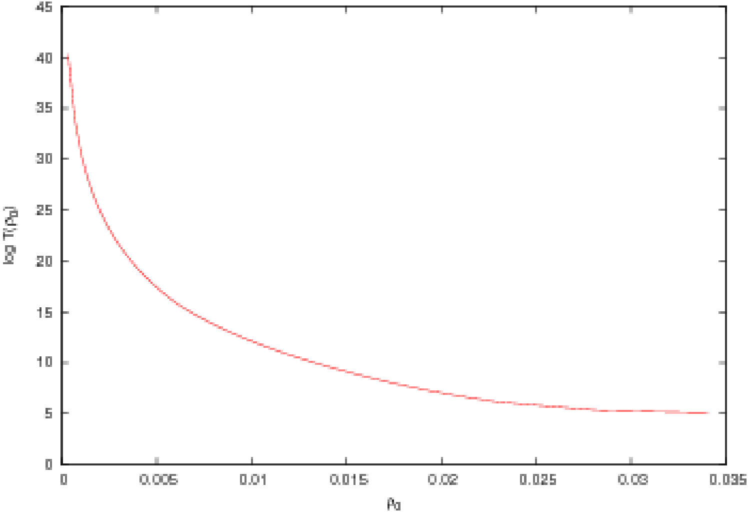
- The "optimal stability time" $T\big(\rho_0, \rho_{opt}(\rho_0, N_{opt}(\rho_0)), N_{opt}(\rho_0)\big)$ depends only on the initial radius ρ_0.

The optimal normalization order



The estimated "stability time" of the secular Hamiltonian



Comments about our results

- We considered a secular Hamiltonian model of the **planar Sun–Jupiter–Saturn–Uranus system**, providing an approximation of the motions of the secular variables *up to order two in the masses*. Our results ensure that such a system is *stable for a time comparable to the age of the universe* just in a domain with a radius that is about **a half of the real distance of the initial secular variables from the origin**.

- We are confident that the reduction of the angular momentum before the "Kolmogorov's like" step will improve significantly our results and this is what we plan to do in a near future.

- If the reduction of the angular momentum won't improve our estimates, we have to introduce some new ideas.