Lower dimensional elliptic tori in planetary systems via normal form

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Based on research works in collaboration with
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Starting point

M. S., U. Locatelli and A. Giorgilli: "*A semi-analytic algorithm for constructing lower dimensional elliptic tori in planetary systems*", Celest. Mech. Dyn. Astr., **111**, 337–361 (2011).

- construction algorithm based on normal form via Lie series;
- explicit computation for a planar model of the SJSU system;
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Fill the gap

- a rigorous proof of the convergence was still lacking: in the present talk we give such a proof.

Hamiltonian of a planetary system

$$F(r,\tilde{r}) = T^{(0)}(\tilde{r}) + U^{(0)}(r) + T^{(1)}(\tilde{r}) + U^{(1)}(r)\,,$$

where r are the heliocentric coordinates and $\tilde{r}$ the conjugated momenta.

$$\begin{aligned}
T^{(0)}(\tilde{r}) &= \frac{1}{2}\sum_{j>0} \|\tilde{r_j}\|^2 \left(\frac{1}{m_0} + \frac{1}{m_j}\right)\,,\\
U^{(0)}(r) &= -\mathcal{G}\sum_{j>0} \frac{m_0 m_j}{\|r_j\|}\,,\\
T^{(1)}(\tilde{r}) &= \sum_{i<j} \frac{\tilde{r_i}\cdot\tilde{r_j}}{m_0}\,,\\
U^{(1)}(r) &= -\mathcal{G}\sum_{i<j} \frac{m_i m_j}{\|r_i - r_j\|}
\end{aligned}$$

The Poincaré variables in the plane

$$\underbrace{\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j}_{\text{fast variables}}$$

$$\underbrace{\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j) \quad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j)}_{\text{secular variables}}$$

where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and perihelion argument of the j-th planet, respectively.

Fixed fast actions

The semi-major axes are constant, up to order two in the masses.
Let us introduce the translated fast actions $L = \Lambda - \Lambda^*$.

Expansion of the Hamiltonian

The translated Hamiltonian reads

$$H^{(\mathcal{T}_F)} = n^* \cdot L + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(L) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(L, \lambda, \xi, \eta)$$

where $h_{j_1,0}^{(Kep)}$ is an homogeneous polynomial of degree j_1 in L and

$$h_{j_1,j_2}^{(\mathcal{T}_F)} \text{is a } \begin{cases} \text{hom. pol. of degree } j_1 \text{ in } L\,, \\ \text{hom. pol. of degree } j_2 \text{ in } (\xi, \eta)\,, \\ \text{with coeff. that are trig. pol. in } \lambda\,. \end{cases}$$

Secular theory

- $L = 0$ is *almost* constant;
- $\lambda(t) \simeq \lambda(0) + n^* t$;
- average over the fast angles (A.-S. Libert, Thursday)

  $H^{(\mathcal{T}_F)}(L, \lambda, \xi, \eta) \rightarrow$ AVERAGE $\rightarrow \mathcal{H}^{(sec)}(\xi, \eta)$;

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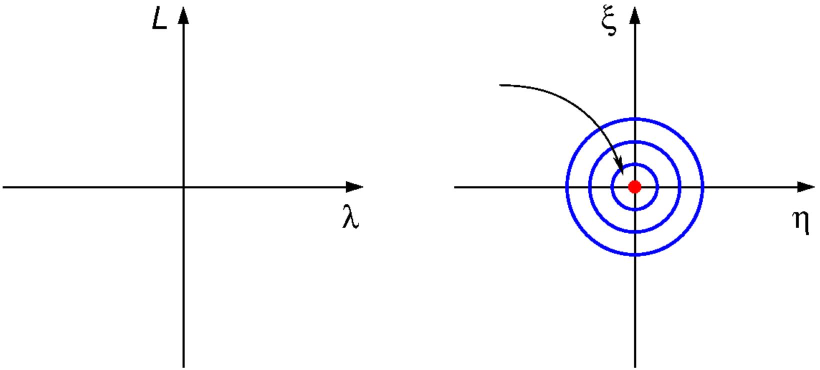

Lower dimensional elliptic torus

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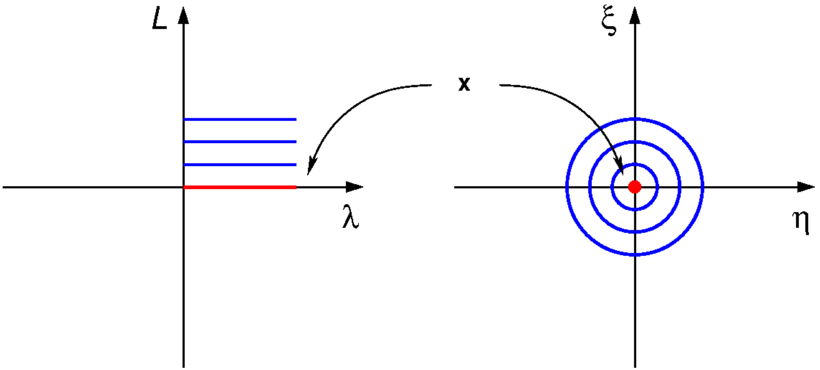
- ~~average over the fast angles~~

  $H^{(\mathcal{T}_F)}(L, \lambda, \xi, \eta) \rightarrow$ **NORMAL FORM** $\rightarrow$ elliptic torus;
- reintroduce the fast dynamics.



State of our main result

We consider the $2(n_1 + n_2)$-dimensional phase space:

$$\big((p, q) \in \mathcal{O}_1 \times \mathbb{T}^{n_1}\big) \;\times\; \big((x, y) \in \mathcal{O}_2 \subseteq \mathbb{R}^{2n_2}\big),$$

where both $\mathcal{O}_1 \subseteq \mathbb{R}^{n_1}$ and $\mathcal{O}_2$ are open sets including the origin.

We also introduce an open set $\mathcal{U} \subset \mathbb{R}^{n_1}$ and the frequency vector $\omega^{(0)} \in \mathcal{U}$ which plays the role of a parameter.

Consider the following family of real Hamiltonians, parameterized by the n_1-dimensional frequency vector $\omega^{(0)}$,

$$\begin{aligned}\mathcal{H}^{(0)}(p, q, x, y; \omega^{(0)}) = \omega^{(0)} \cdot p + \varepsilon \sum_{j=1}^{n_2} \left[\frac{\Omega_j^{(0)}(\omega^{(0)})}{2} \left(x_j^2 + y_j^2\right) \right] \\ + \varepsilon \mathcal{F}_0(q; \omega^{(0)}) + \varepsilon \mathcal{F}_1(q, x, y; \omega^{(0)}) + \varepsilon \mathcal{F}_2(p, q, x, y; \omega^{(0)}) \\ + \mathcal{F}_{\mathrm{h.o.t.}}(p, q, x, y; \omega^{(0)}) \;,\end{aligned}$$

with ε playing the usual role of small parameter.

Let us assume that:

(a) the frequencies $\Omega_j^{(0)}$ and the functions $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_{\rm h.o.t.}$ are analytic functions of $(p, q, x, y; \omega^{(0)}) \in \mathcal{O}_1 \times \mathbb{T}^{n_1} \times \mathcal{O}_2 \times \mathcal{U}$;

(b) one has $\Omega_i^{(0)}(\omega^{(0)}) \neq \Omega_j^{(0)}(\omega^{(0)})$ for $\omega^{(0)} \in \mathcal{U}$ and $1 \le i < j \le n_2$;

(c) $\mathcal{F}_0$ is independent of p and (x, y) ;
$\mathcal{F}_1$ is independent of p and linear in (x, y) ;
$\mathcal{F}_2$ is either linear in p or quadratic in (x, y) ;
$\mathcal{F}_{\rm h.o.t.} = o\big(\|p\| + \|(x, y)\|^2\big)$;

(d) $\mathcal{F}_{\rm h.o.t.}(p, q, x, y; \omega^{(0)}) = \mathcal{F}_{\rm int}(p; \omega^{(0)}) + \varepsilon \mathcal{F}_{\rm n.i.}(p, q, x, y; \omega^{(0)})$;
moreover, the average of $\mathcal{F}_2$ over the angles is equal to zero;

(e) $\mathcal{H}^{(0)}$ is invariant with respect to the θ-family of canonical diffeomorphisms

$$
\begin{aligned}
&\big(p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_1}, x_1, \ldots, x_{n_2}, y_1, \ldots, y_{n_2}\big) \mapsto \\
&\quad \big(p_1, \ldots, p_{n_1}, q_1 + \vartheta, \ldots, q_{n_1} + \vartheta, \\
&\qquad x_1 \cos\vartheta + y_1 \sin\vartheta, \ldots, x_{n_2} \cos\vartheta + y_{n_2} \sin\vartheta\big) \\
&\qquad y_1 \cos\vartheta - x_1 \sin\vartheta, \ldots, y_{n_2} \cos\vartheta - x_{n_2} \sin\vartheta\big)
\end{aligned}
$$

where $\vartheta \in \mathbb{T}$;

(f) one has

$$
\begin{aligned}
\sup \left|\mathcal{F}_j(p, q, x, y; \omega^{(0)})\right| &\leq E \\
\sup \left|\mathcal{F}_{\mathrm{h.o.t.}}(p, q, x, y; \omega^{(0)})\right| &\leq E \ ,
\end{aligned}
$$

for $j = 0,\ 1,\ 2$ and for some $E > 0$.

Then, there is a positive $\varepsilon^{\star}$ such that for $0 \le \varepsilon < \varepsilon^{\star}$ the following statement holds true:

there exists a non-resonant set $\mathcal{U}^{(\infty)} \subset \mathcal{U}$ of positive Lebesgue measure, such that for each $\omega^{(0)} \in \mathcal{U}^{(\infty)}$ there exists an analytic canonical transformation $(p,q,x,y) = \psi^{(\infty)}_{\omega^{(0)}}(P,Q,X,Y)$ leading the Hamiltonian in the normal form

$$\begin{aligned}\mathcal{H}^{(\infty)}(P,Q,X,Y;\omega^{(0)}) =& \omega^{(\infty)} \cdot P + \sum_{j=1}^{n_2} \varepsilon\, \Omega_j^{(\infty)} \left(\frac{X_j^2 + Y_j^2}{2}\right) \\ & + o\big(\|P\| + \|(X,Y)\|^2\big)\ ,\end{aligned}$$

where $\omega^{(\infty)} = \omega^{(\infty)}(\omega^{(0)})$ and $\Omega^{(\infty)} = \Omega^{(\infty)}(\omega^{(0)})$.

Lower dimensional elliptic torus

The torus $P = 0$, $X = Y = 0$ is clearly invariant and elliptic, and carries a quasi-periodic motion with frequencies $\omega^{(\infty)}$.

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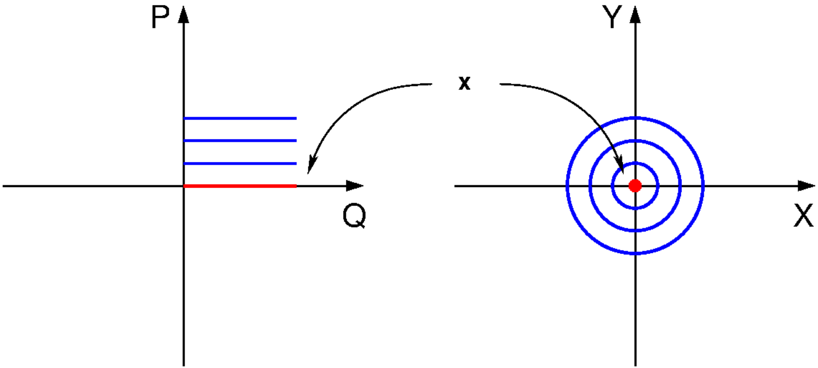

Main idea

- If $\mathcal{F}_0 = \mathcal{F}_1 = \mathcal{F}_2 = 0$, then the Hamiltonian is in normal form.
- $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ should be killed.
- We aim to prove that an elliptic torus, possibly with different frequencies, persists under the perturbation.

Main idea

- If $\mathcal{F}_0 = \mathcal{F}_1 = \mathcal{F}_2 = 0$, then the Hamiltonian is in normal form.
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Remarks

- Setting $\boldsymbol{\varepsilon = 0}$ one is left with the so-called **"keplerian approximation"**, depending on the actions p only.
- The frequencies of the oscillations transversal to an elliptic torus are $\mathcal{O}(\varepsilon)$ with respect to those related to the quasi-periodic motion on the invariant lower dimensional torus.
- Natural distinction between the **fast variables** (p, q) and the slow **secular variables** (x, y) .
- Hypothesis (e) is natural if one is interested in planetary systems. Actually, it means that $\mathcal{H}^{(0)}$ is invariant with respect to rotations around the direction of the total angular momentum. **This is equivalent to assume the so-called "d'Alembert rules"**.

Technical tools

We introduce the complex variables

$$z = (x + \mathbf{i}y)/\sqrt{2} \ .$$

For some fixed positive integer K we introduce the classes of functions $\widehat{\mathcal{P}}_{\hat{m},\hat{l},sK}$ with integers $\hat{m},\ \hat{l},\ s \geq 0$, which can be written as

$$g(p,q,z,\mathbf{i}\bar{z}) = \sum_{\substack{m\in\mathbb{N}^{n_1} \\ |m|=\hat{m}}} \ \sum_{\substack{(l,\bar{l})\in\mathbb{N}^{2n_2} \\ |l|+|\bar{l}|=\hat{l}}} \ \sum_{\substack{k\in\mathbb{Z}^{n_1} \\ |k|\leq sK}} c_{m,l,\bar{l},k}\, p^m z^l (\mathbf{i}\bar{z})^{\bar{l}} \exp(\mathbf{i}k\cdot q) \ ,$$

with coefficients $c_{m,l,\bar{l},k} \in \mathbb{C}$.

Technical tools

Furthermore we say that $g \in \mathcal{P}_{\ell,sK}$ in case

$$g \in \bigcup_{\substack{\hat{m}\geq 0\,,\,\hat{l}\geq 0\\ 2\hat{m}+\hat{l}=\ell}} \widehat{\mathcal{P}}_{\hat{m},\hat{l},sK}$$

and the Taylor-Fourier expansion of g satisfies the following property: setting

$$\mathcal{C}_{\mathcal{M}}(l,\bar{l}) = \sum_{j=1}^{n_2}(l_j - \bar{l}_j)\ , \qquad \mathcal{C}_{\mathcal{I}}(k) = \sum_{j=1}^{n_1} k_j\ ,$$

one has $c_{m,l,\bar{l},k} = 0$ for $\mathcal{C}_{\mathcal{M}}(l,\bar{l}) \neq \mathcal{C}_{\mathcal{I}}(k)$. We also set $\mathcal{P}_{-2,sK} = \mathcal{P}_{-1,sK} = \{0\}$ for $s \geq 0$, $K > 0$.

Finally we shall denote by

$$\langle g \rangle_{\vartheta} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g\ \mathrm{d}\vartheta_1 \ldots \mathrm{d}\vartheta_n\ ,$$

the average of a function g with respect to the angles ϑ.

Initial Hamiltonian

$$H^{(0)} = \omega^{(0)} \cdot p + \varepsilon \sum_{j=1}^{n_2} \Omega_j^{(0)} z_j \bar{z}_j + \sum_{\ell > 2} \sum_{s \geq 0} \varepsilon^s f_\ell^{(0,s)}$$

$$+ \sum_{s \geq 1} \varepsilon^s f_0^{(0,s)} + \sum_{s \geq 1} \varepsilon^s f_1^{(0,s)} + \sum_{s \geq 1} \varepsilon^s f_2^{(0,s)} \ ,$$

where $f_\ell^{(0,s)} \in \mathcal{P}_{\ell, sK}$ and, as in hypothesis (c) of the theorem, $f_\ell^{(0,0)} = f_\ell^{(0,0)}(p; \omega^{(0)})$ for $\ell \geq 3$ and $\langle f_2^{(0,1)} \rangle_q = 0$.

Normal form approach

In the spirit of the original Kolmogorov's proof scheme, we construct an infinite sequence of Hamiltonians $\left\{H^{(r)}\right\}_{r\geq 0}$ with the request that each $H^{(r)}$ is in normal form up to order r (in a sense to be defined later).

We perform a sequence of normalization steps, transforming the Hamiltonian $H^{(r-1)}$ into $H^{(r)}$ via a near the identity canonical transformation,

$$\mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}} \circ \exp\left(\varepsilon^r \mathcal{L}_{\chi_2^{(r)}}\right) \circ \exp\left(\varepsilon^r \mathcal{L}_{\chi_1^{(r)}}\right) \circ \exp\left(\varepsilon^r \mathcal{L}_{\chi_0^{(r)}}\right)$$

where $\mathcal{L}_g \cdot = \{\cdot, g\}$ is the Lie derivative operator and $\chi_0^{(r)}(q) \in \mathcal{P}_{0,rK}$, $\chi_1^{(r)}(q, z, \mathbf{i}\bar{z}) \in \mathcal{P}_{1,rK}$, $\chi_2^{(r)}(p, q, z, \mathbf{i}\bar{z}) \in \mathcal{P}_{2,rK}$.

The Lie transform operator $\mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}}$ actually induces a canonical linear change of the coordinates $(z, \mathbf{i}\bar{z})$.

Small divisors

The main difference with respect to the original Kolmogorov's algorithm is that the frequencies $\omega^{(r)}$ and $\Omega^{(r)}$ may change at every normalization step by a small quantity

In order to control the small divisors, we need to introduce at each r-th step two non-resonance conditions up to a finite order rK, namely

$$\min_{\substack{k\in\mathbb{Z}^{n_1}\,,\,0<|k|\le rK\\ l\in\mathbb{Z}^{n_2}\,,\,0\le|l|\le 2}} \left|k\cdot\omega^{(r-1)}(\omega^{(0)})+\varepsilon\, l\cdot\Omega^{(r-1)}(\omega^{(0)})\right| \ge a_r\ ,$$

and

$$\min_{1\le i<j\le n_2} \left|\Omega_i^{(r-1)}(\omega^{(0)})-\Omega_j^{(r-1)}(\omega^{(0)})\right| \ge b_r\ ,$$

where $\{a_r\}_{r\ge1}$ and $\{b_r\}_{r\ge1}$ are two monotonically decreasing sequences such that $a_r\to0$ and $b_r\to b_\infty>0$ when $r\to+\infty$.

Normal form up to order $r-1$

$$H^{(r-1)} = \omega^{(r-1)} \cdot p + \varepsilon \sum_{j=1}^{n_2} \Omega_j^{(r-1)} z_j \bar{z}_j + \sum_{\ell>2} \sum_{s\geq 0} \varepsilon^s f_\ell^{(r-1,s)}$$
$$+ \sum_{s\geq r} \varepsilon^s f_0^{(r-1,s)} + \sum_{s\geq r} \varepsilon^s f_1^{(r-1,s)} + \sum_{s\geq r} \varepsilon^s f_2^{(r-1,s)} \ ,$$

where $f_\ell^{(r-1,s)} \in \mathcal{P}_{\ell,sK}$; moreover, we have $f_\ell^{(r-1,0)} = f_\ell^{(r-1,0)}(p;\omega^{(0)})$ for $\ell \geq 3$ and, just for $r=1$, $\langle f_2^{(0,1)} \rangle_q = 0$.

First stage

$$\mathcal{L}_{\chi_0^{(r)}}\left(\omega^{(r-1)}\cdot p\right)+f_0^{(r-1,r)}-\langle f_0^{(r-1,r)}\rangle_q=0\ .$$

Second stage

$$\mathcal{L}_{\chi_1^{(r)}}\left(\omega^{(r-1)}\cdot p+\varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r-1)}z_j\bar{z}_j\right)+f_1^{(\mathrm{I};r,r)}=0\ .$$

Third stage

$$\mathcal{L}_{\chi_2^{(r)}}\left(\omega^{(r-1)}\cdot p+\varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r-1)}z_j\bar{z}_j\right)+f_2^{(\mathrm{II};r,r)}-\langle f_2^{(\mathrm{II};r,r)}\rangle_q=0\ .$$

Quadratic part

$$\langle f_2^{(\mathrm{II};r,r)} \rangle_q = \sum_{|m|=1} c_{m,0,0,0}^{(\mathrm{II};r)} p^m + \sum_{|l|=|\bar{l}|=1} c_{0,l,\bar{l},0}^{(\mathrm{II};r)} z^l (\mathbf{i}\bar{z})^{\bar{l}} \ .$$

Diagonalization of the quadratic part

$$\mathcal{T}_{\mathcal{D}_2^{(r)}} \left(\varepsilon Z_0^{(r)} \right) + \mathcal{T}_{\mathcal{D}_2^{(r)}} \left(\varepsilon^r g_1^{(r)} \right) = \sum_{j=0}^{+\infty} \varepsilon^{j(r-1)+1} Z_j^{(r)} \ ,$$

where

$$Z_0^{(r)} = \sum_{j=1}^{n_2} \Omega_j^{(r-1)} z_j \bar{z}_j \ , \qquad g_1^{(r)}(z, \mathbf{i}\bar{z}) = f_2^{(\mathrm{III};r,r)}(0, z, \mathbf{i}\bar{z})$$

and $Z_j^{(r)}$, for $j \geq 1$, is the polynomial

$$Z_j^{(r)} = \sum_{|l|=1} c_{0,l,l,0}^{(r;j)} z^l (\mathbf{i}\bar{z})^l \ ,$$

with coefficients $c_{0,l,l,0}^{(r;j)}$ to be found.

Diagonal part

$f_2^{(\mathrm{III};r,r)}$ still contains a part that is $\mathcal{O}(\varepsilon^r)$ and belongs to $\mathcal{P}_{2,0}$, i.e., $\sum_{|m|=1} c_{m,0,0,0}^{(\mathrm{II};r)} p^m + \sum_{i\geq 1} \varepsilon^{i(r-1)} \sum_{|l|=1} c_{0,l,l,0}^{(r;i)} z^l (\mathbf{i}\bar{z})^l$.

Change of the frequencies

$$\omega_j^{(r)} = \omega_j^{(r-1)} + \varepsilon^r \frac{\partial f_2^{(\mathrm{III};r,r)}}{\partial p_j} \qquad \text{for } j = 1, \ldots, n_1$$

and

$$\Omega_j^{(r)} = \Omega_j^{(r-1)} + \sum_{i=1}^{+\infty} \left[\varepsilon^{i(r-1)} \frac{\partial^2 Z_i^{(r)}}{\partial z_j \partial(\mathbf{i}\bar{z}_j)} \right] \qquad \text{for } j = 1, \ldots, n_2 \ .$$

This conclues the r-th normalization step!

Analytical setting

We introduce the complex domains $\mathcal{D}_{\rho,R,\sigma,h} = \mathcal{G}_\rho \times \mathbb{T}^{n_1}_\sigma \times \mathcal{B}_R \times \mathcal{W}_h$, where $\mathcal{G}_\rho \subset \mathbb{C}^{n_1}$ and $\mathcal{B}_R \subset \mathbb{C}^{n_2} \times \mathbb{C}^{n_2}$ are open balls centered at the origin with radii ρ and R, respectively, $\mathcal{W}$ is a subset of $\mathbb{R}^{n_1}$ while the subscripts σ and h denote the usual complex extensions

Definition

We say that the sequence $\{a_r\}_{r\geq 1}$ satisfies the **condition** $\boldsymbol{\tau}$, if

$$-\sum_{r\geq 1} \frac{\log a_r}{r(r+1)} = \Gamma < \infty \ .$$

Remark

The **condition** $\boldsymbol{\tau}$ turns out to be equivalent to the Bruno's one, but it produces better analytical estimates.

Analytical part

If parameter ε is smaller than the "analytical threshold value" $\varepsilon_{\rm an}^{\star}$, being

$$\varepsilon_{\rm an}^{\star} = \frac{1}{2^8}\left(\frac{\min\{1\,,\,\bar{b}\}}{\mathcal{A}}\right)^2 \qquad \text{with} \qquad \mathcal{A} = \left(2^{18} M\, e^{\Gamma}\right)^3 \ .$$

Then, there exists an analytic canonical transformation $\Phi_{\omega^{(0)}}^{(\infty)}$ such that the Hamiltonian $H^{(\infty)} = H^{(0)} \circ \Phi_{\omega^{(0)}}^{(\infty)}$ is in normal form,

$$H^{(\infty)} = \omega^{(\infty)} \cdot p + \varepsilon \sum_{j=1}^{n_2} \Omega_j^{(\infty)} z_j \bar{z}_j + \sum_{\ell>2} \sum_{s\ge 0} \varepsilon^s f_\ell^{(\infty,s)} \ .$$

The limit values of the frequency vectors $\omega^{(\infty)}$ and $\Omega^{(\infty)}$ are well defined, being $\{\omega^{(r)}\}_{r\ge 0}$ and $\{\Omega^{(r)}\}_{r\ge 0}$ Cauchy sequences,

$$\max_{1\le i\le n_1}\left\{\left|\omega_i^{(r)} - \omega_i^{(r-1)}\right|\right\} \le \sigma\left(\varepsilon\mathcal{A}\right)^r \ ,$$
$$\max_{1\le j\le n_2}\left\{\left|\Omega_j^{(r)} - \Omega_j^{(r-1)}\right|\right\} \le \varepsilon^{r-1}\mathcal{A}^r \ ,$$

Warning!

We need to ensure that the non-resonance conditions are satisfied at each normalization step!

Goal

We want to show that the set of initial frequencies to which our algorithm applies has relative big measure.

Thanks to a lemma...

The set of frequencies, $\mathcal{W}$, is *non-resonant* up to the finite order $2K$, namely every $\omega^{(0)} \in \mathcal{W}$ satisfy

$$\min_{\substack{k\in\mathbb{Z}^{n_1}\,,\,0<|k|\leq 2K\\ l\in\mathbb{Z}^{n_2}\,,\,0\leq|l|\leq 2}} \left|k\cdot\omega^{(0)}+\varepsilon\, l\cdot\Omega^{(0)}(\omega^{(0)})\right| > \frac{2\gamma}{K^{\tau}}$$

and

$$\min_{1\leq i<j\leq n_2} \left|\Omega_i^{(0)}(\omega^{(0)})-\Omega_j^{(0)}(\omega^{(0)})\right| > 2\bar{b}\ .$$

The Jacobian of $\Omega^{(0)}(\omega^{(0)})$ is uniformly bounded in the extended domain $\mathcal{W}_{h_0}$, namely

$$\left|\partial\Omega^{(0)}/\partial\omega^{(0)}\right|_{\infty;\mathcal{W}_{h_0}} \leq J_0 < \infty\ .$$

Restrictions of the frequencies domain

Let $\mathcal{W}^{(0)} \subset \mathbb{R}^{n_1}$ be the starting domain and $\mathcal{W}^{(0)}_{h_0}$ its complex extension.

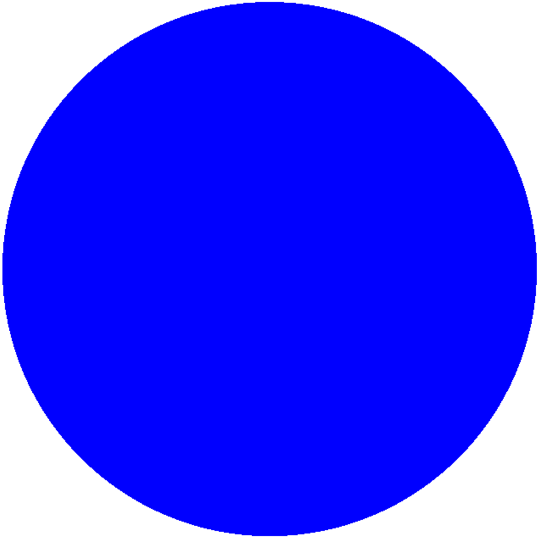
We consider a sequence of complex extended domains $\mathcal{W}^{(0)}_{h_0} \supseteq \mathcal{W}^{(1)}_{h_1} \supseteq \mathcal{W}^{(2)}_{h_2} \supseteq \ldots$, with $\{h_r\}_{r\geq 0}$ a positive non-increasing sequence of real numbers such that $\omega^{(r)}(\omega^{(0)})$ admits an inverse function $\varphi^{(r)}$ well defined on $\mathcal{W}^{(r)}_{h_r}$.

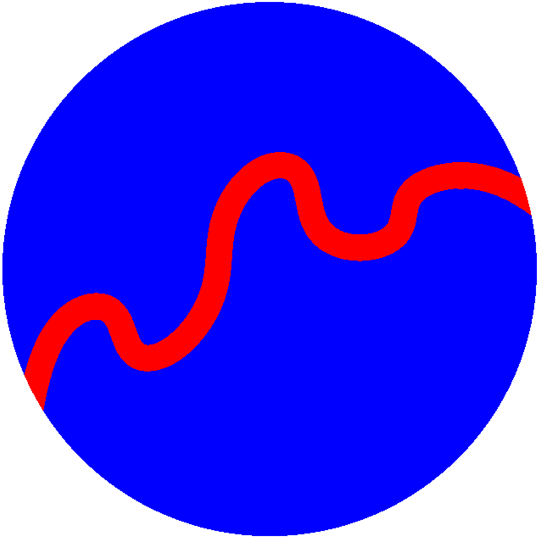
We remove from $\mathcal{W}^{(r-1)}$ all the resonant regions related to the new small divisors appearing in the formal algorithm,

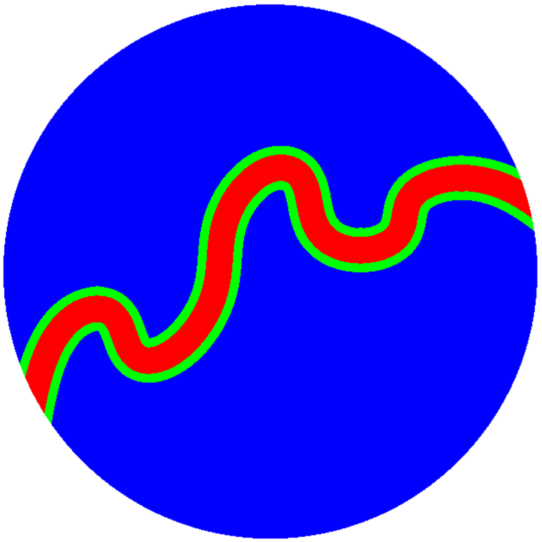
$$\mathcal{W}^{(r)} = \mathcal{W}^{(r-1)} \setminus \mathcal{R}^{(r)} , \qquad \text{with} \quad \mathcal{R}^{(r)} = \bigcup_{\substack{rK<|k|\leq(r+1)K \\ |l|\leq 2}} \mathcal{R}^{(r)}_{k,l} ,$$

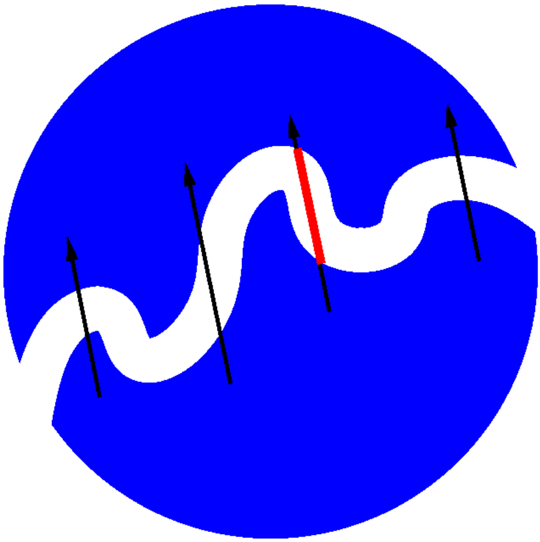
being

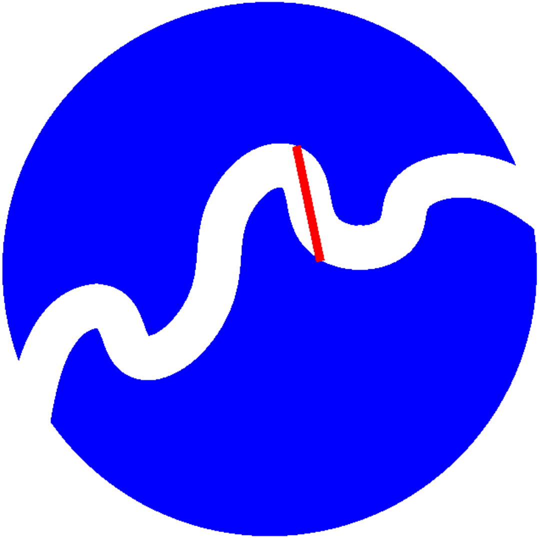
$$\mathcal{R}^{(r)}_{k,l} = \left\{ \omega \in \mathcal{W}^{(r-1)} : \left| k\cdot\omega + \varepsilon l \cdot \Omega^{(r)} \circ \varphi^{(r)}(\omega) \right| < \frac{2\gamma}{((r+1)K)^{\tau}} \right\} .$$

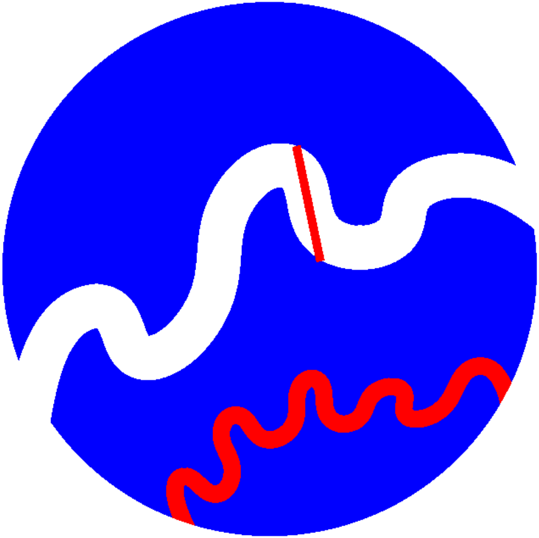


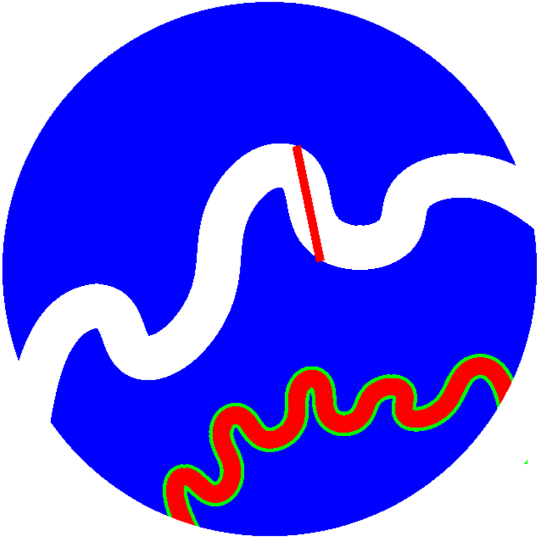


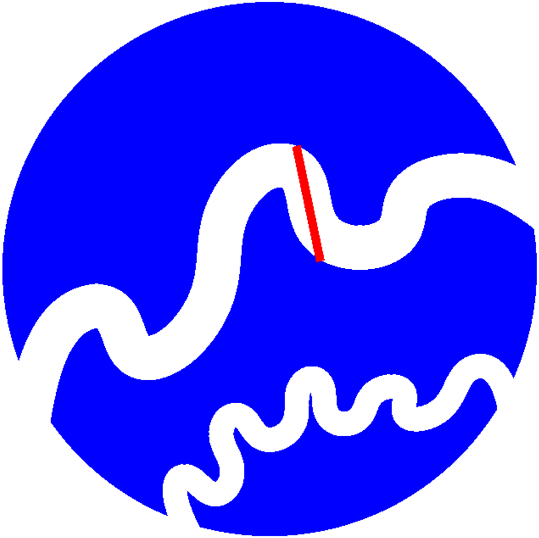


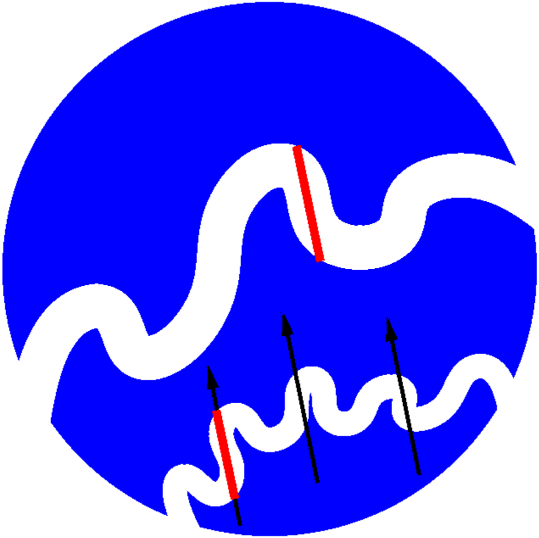


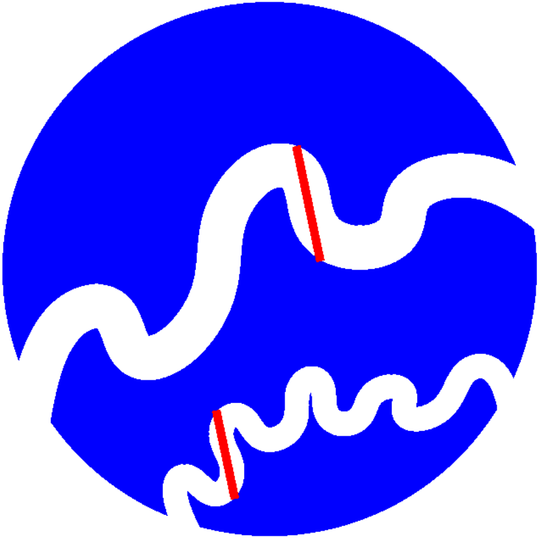


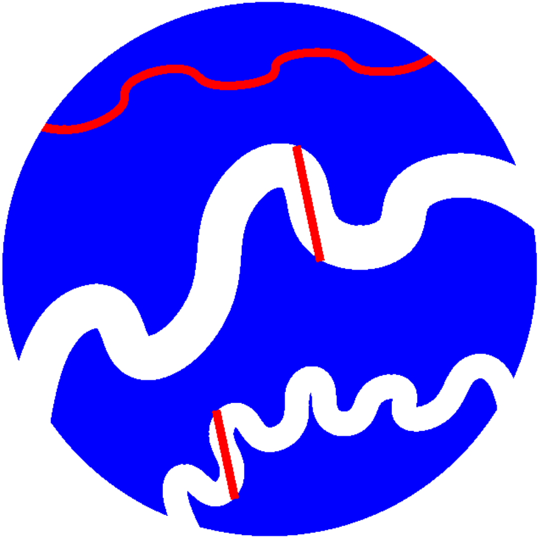


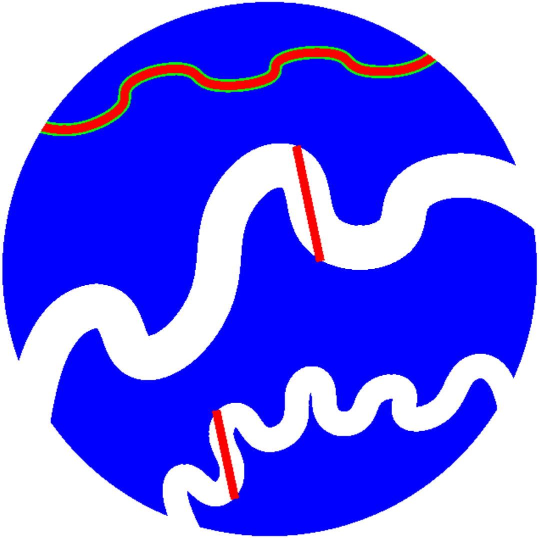


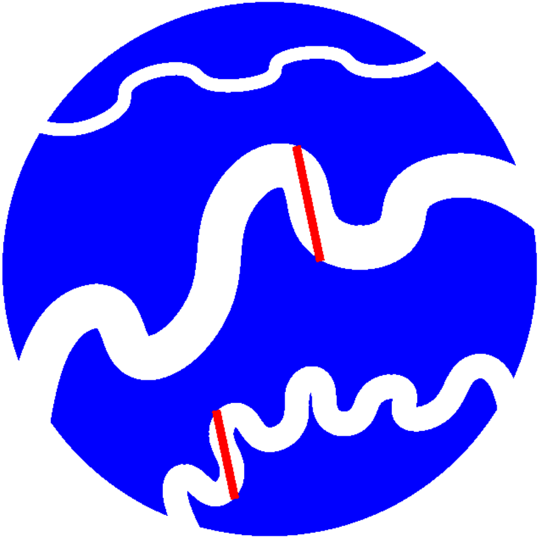


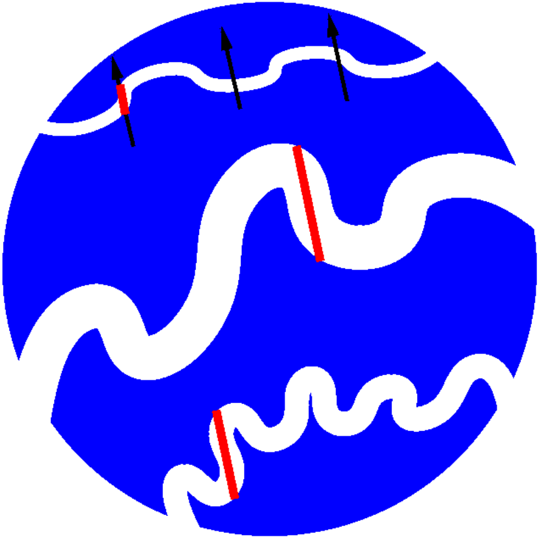


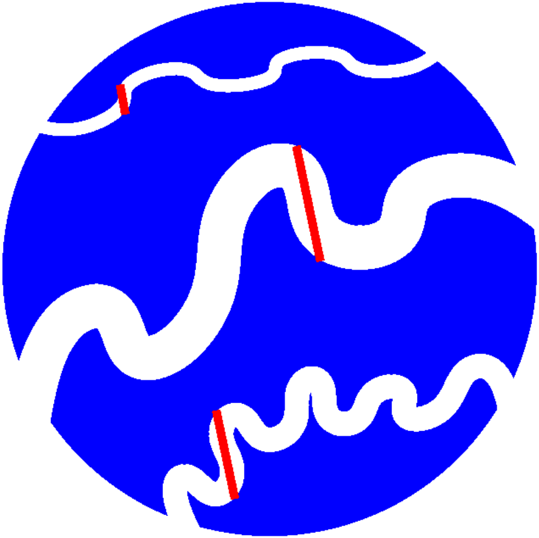












Geometric part

Define the sequence $\{h_r\}_{r\geq 0}$ of complex extensions as

$$h_0 = \min\left\{\frac{\gamma\eta}{4K^\tau}\,,\ \frac{\bar{b}}{4J_0}\right\} \qquad \text{and} \qquad h_r = \frac{h_{r-1}}{2^{\tau+2}} \ \text{ for } r \geq 1\ ,$$

where $\eta = \min\{1/K\,,\,\sigma\}\,.$

The function $\omega^{(s)}(\omega^{(0)})$ has analytic inverse $\varphi^{(s)}$ on $\mathcal{W}^{(s)}_{h_s}$, for $0 \leq s \leq r-1$, where $\varphi^{(0)} = \mathrm{Id}$

If parameter ε is smaller than the "geometric threshold value"

$$\varepsilon^*_{\mathrm{ge}} = \min\left\{\frac{1}{(2J_0+1)\eta}\,,\ \frac{1}{2^{\tau+3}\mathcal{A}}\,\min\left\{1\,,\,\frac{h_0}{8\mathcal{A}}\,,\,\frac{\bar{b}}{8\mathcal{A}}\right\}\right\}\ ;$$

Geometric part

Then the following non-resonance inequalities hold true:

$$\min_{\substack{k\in\mathbb{Z}^{n_1}\,,\,0<|k|\le(r+1)K\\ l\in\mathbb{Z}^{n_2}\,,\,0\le|l|\le 2}} \inf_{\omega\in\mathcal{W}_{h_r}^{(r)}} \left| k\cdot\omega+\varepsilon l\ \cdot\Omega^{(r)}\big(\varphi^{(r)}(\omega)\big)\right| \ge \frac{\gamma}{\big((r+1)K\big)^{\tau}}\ ,$$

$$\min_{1\le i<j\le n_2} \inf_{\omega\in\mathcal{W}_{h_r}^{(r)}} \left|\Omega_i^{(r)}\big(\varphi^{(r)}(\omega)\big)-\Omega_j^{(r)}\big(\varphi^{(r)}(\omega)\big)\right| \ge \bar{b}\ .$$

Moreover, the Lipschitz constant related to the Jacobian of the functions $\varphi^{(r)}$ and $\Omega^{(r)}\circ\varphi^{(r)}$ are uniformly bounded as

$$\left|\frac{\partial\big(\varphi^{(r)}-\mathrm{Id}\big)}{\partial\omega}\right|_{\infty;\mathcal{W}_{h_r}^{(r)}} \le \varepsilon\sigma\ , \qquad \left|\frac{\partial\big(\Omega^{(r)}\circ\varphi^{(r)}\big)}{\partial\omega}\right|_{\infty;\mathcal{W}_{h_r}^{(r)}} \le 2J_0+1\ .$$

To conclude...

We now need to estimate the volume of the resonant zones that we must remove at each step of the procedure and show that the final “good domain”, $\lim_{r\to\infty}\varphi^{(r)}(\mathcal{W}^{(r)})$, has positive Lebesgue measure.

Technical tools

We denote by $\mathcal{K}_l^{(r)}$ the closed convex hull of the gradient set

$$\mathcal{G}_l^{(r)} = \Big\{ \partial_\omega \big[\varepsilon l \cdot \Omega^{(r)}\big(\varphi^{(r)}(\omega)\big)\big] \ : \ \omega \in \mathcal{W}^{(r)} \Big\} \ .$$

Under the assumption $\varepsilon < 1/(4J_0 + 1)$,

$$\mathrm{dist}(k, \mathcal{K}_l^{(r)}) \geq 1/2 \quad \text{for } k \in \mathbb{Z}^{n_1} \setminus \{0\} \ .$$

Lemma (Pöschel 1989)

If $\mathrm{dist}(k, \mathcal{K}_l^{(r)}) = s > 0$ then

$$\mathrm{m}(\mathcal{R}_{k,l}^{(r)}) \leq \frac{4\gamma}{((r+1)K)^\tau} \frac{D^{n_1 - 1}}{s} \ ,$$

where D is the diameter of $\mathcal{W}^{(0)}$ with respect to the sup-norm.

Volume of the resonant region

For $k \in \mathbb{Z}^{n_1} \setminus \{0\}$ we have

$$\mathrm{m}\Big(\varphi^{(r)}\big(\mathcal{R}_{k,l}^{(r)}\big)\Big) \le \frac{8\gamma D^{n_1-1}}{((r+1)K)^{\tau}} \sup_{\omega \in \mathcal{W}^{(r)}} \det\left(\frac{\partial \varphi^{(r)}}{\partial \omega}\right) .$$

Using the Gershgorin circle theorem, we get the bound

$$\sup_{\omega \in \mathcal{W}^{(r)}} \det\left(\frac{\partial \varphi^{(r)}}{\partial \omega}\right) \le 2 \qquad \text{when} \quad \varepsilon \le \frac{\log 2}{\sigma n_1^2} .$$

$$\sum_{r=2}^{\infty} \sum_{\substack{rK<|k|\le(r+1)K \\ |l|\le 2}} \mathrm{m}\Big(\varphi^{(r)}\big(\mathcal{R}_{k,l}^{(r)}\big)\Big) \le \gamma \, \frac{2^{n_1+4} c_{n_2} D^{n_1-1}}{K^{\tau-n_1}} \sum_{r=3}^{\infty} \frac{1}{r^{\tau-n_1+1}} ,$$

where $c_{n_2} = (2n_2+2)(2n_2+1)/2$.

The last series is convergent if $\tau > n_1$ and it is of order $\mathcal{O}(\gamma)$.

Summing up...

If the smaller parameter ε is such that $\varepsilon < \varepsilon^\star$, with

$$\varepsilon^\star = \min\left\{\varepsilon^\star_{\mathrm{an}}\,,\,\varepsilon^\star_{\mathrm{ge}}\,,\,\frac{1}{4(2J_0+1)}\,,\,\frac{\log 2}{\sigma n_1^2}\right\}\ ,$$

than both the "analytical part" and the "geometric part" holds true.

Combining the two previous results we conclude the proof.

Thanks for your attention!

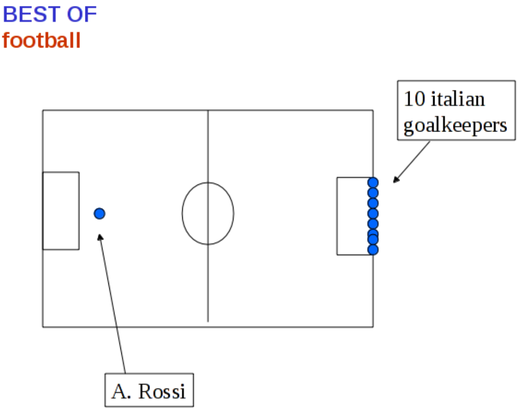
Thanks for your attention!

Four years ago...

Italy (+Catalunya)	**5**
Rest of the World	**10**

...Thursday is coming!

Thanks for your attention!



Four years ago...
Italy (+Catalunya) 5
Rest of the World 10
...Thursday is coming!

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