Third La Plata International School on Astronomy and Geophysics

Chaos, diffusion and non-integrability in Hamiltonian Systems: Applications to Astronomy

Marco Sansottera: introduction to Algebraic Manipulators, representation of polynomials and trigonometric polynomials using indexing functions. Applications to the Hénon-Heiles problem and triangular libration points. Exponential stability estimates.

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1

INDEXING FUNCTIONS

Here I describe the algorithms providing an indexing method for integer vectors, i.e., subsets of $\mathbf{Z}^n$.

The basic remark for constructing an index function is the following. Suppose that we are given a countable set $\mathcal{A}$. Suppose also that $\mathcal{A}$ is equipped with a relation of complete ordering, that I will denote by the symbols $\prec$, $\preceq$, $\succ$ and $\succeq$. So, for any two elements $a, b \in \mathcal{A}$ exactly one of the relations $a \prec b$, $a = b$ and $a \succ b$ is true. Suppose also that there is a minimal element in $\mathcal{A}$, i.e., there is $a_0 \in \mathcal{A}$ such that $a \succ a_0$ for all $a \in \mathcal{A}$ such that $a \neq a_0$. Then an index function I is naturally defined as

$$I(a) = \#\{b \in \mathcal{A} \,:\, b \prec a\} \ . \qquad (1.1)$$

If $\mathcal{A}$ is a finite set containing N elements, then $I(\mathcal{A}) = \{0, 1, \ldots, N-1\}$. If $\mathcal{A}$ is an infinite (but countable) set, then $I(\mathcal{A}) = \mathbf{Z}_+$, the set of non negative integers. For instance, the trivial case is $\mathcal{A} = \mathbf{Z}_+$ equipped with the usual ordering relation. In such a case the indexing function is just the identity.

Having defined the function $I(a)$, we are interested in performing the following basic operations:

(i) for a given $a \in \mathcal{A}$ find the index $I(a)$;
(ii) for a given $a \in \mathcal{A}$, find the element next (or prior) to a, if it exists;
(iii) for a given $l \in I(\mathcal{A})$, find $I^{-1}(l)$, i.e., the element $a \in \mathcal{A}$ such that $I(a) = l$.

The problem here is to find an effective way of constructing such a kind of information for some particular subsets of $\mathbf{Z}^n$ that we are interested in. In order to avoid confusions, I shall use the symbols $\prec$, $\preceq$, $\succ$ and $\succeq$ when dealing with an ordering relation in the subset of $\mathbf{Z}^n$ under consideration. The symbols $<$, $\leq$, $\geq$ and $>$ will always denote the usual ordering relation between integers.

As a first elementary example, let us consider the case $\mathcal{A} = \mathbf{Z}$. The usual ordering relation $<$ is useless, because there is no minimal element. However, we can construct a different ordering satisfying our requests as follows.

Let $k, k' \in \mathbf{Z}$. I shall say that $k' \prec k$ in case one of the following relations is true:
(i) $|k'| < |k|$;

(ii) $|k'| = |k| \;\wedge\; k' > k$.

The resulting order is $0, 1, -1, 2, -2, \ldots$, so that 0 is the minimal element.

Constructing the indexing function in this case is easy. Indeed, we have

$$(1.2)\qquad I(0) = 0\ ,\quad I(a) = \begin{cases} 2a - 1 & \text{for } a > 0\ , \\ -2a & \text{for } a < 0\ . \end{cases}$$

The inverse function is also easily constructed:

$$(1.3)\qquad I^{-1}(0) = 0\ ,\quad I^{-1}(l) = \begin{cases} (l+1)/2 & \text{for } l \text{ odd}\ , \\ -l/2 & \text{for } l \text{ even}\ . \end{cases}$$

1.1 The polynomial case

Let us first take $\mathcal{A}_n = \mathbf{Z}_+^n$, i.e., integer vectors with non negative components; formally

$$\mathcal{A}_n = \{k = (k_1, \ldots, k_n) \in \mathbf{Z}^n \;:\; k_1 \geq 0, \ldots, k_n \geq 0\}\ .$$

The index n in $\mathcal{A}_n$ denotes the dimension of the space. This case is named "polynomial" because it occurs precisely in the representation of homogeneous polynomials, and so also in the Taylor expansion of a function of n variables: the integer vectors $k \in \mathcal{A}_n$ represent all possible exponents.

I shall denote by $|k| = k_1 + \ldots + k_n$ the length (or norm) of the vector $k \in \mathbf{Z}_+^n$. Furthermore, to a given vector $k = (k_1, \ldots, k_n) \in \mathcal{A}_n$ I shall associate the vector $t(k) \in \mathcal{A}_{n-1}$ (the *tail* of k) defined as $t(k) = (k_2, \ldots, k_n)$. This definition is meaningful only if $n > 1$, of course.

1.1.1 Ordering relation

Pick a fixed n, and consider the finite family of sets $\mathcal{A}_1 = \mathbf{Z}_+, \ldots, \mathcal{A}_n = \mathbf{Z}_+^n$.

Let $k, k' \in \mathcal{A}_m$, with any $1 \leq m \leq n$. I shall say that $k' \prec k$ in case one of the following conditions is true:

(i) $m \geq 1 \;\wedge\; |k'| < |k|$;
(ii) $m > 1 \;\wedge\; |k'| = |k| \;\wedge\; k_1' > k_1$;
(iii) $m > 1 \;\wedge\; |k'| = |k| \;\wedge\; k_1' = k_1 \;\wedge\; t(k') \prec t(k)$.

In table 1.1 the ordering resulting from this definition is illustrated for the cases $n = 2, 3, 4, 5$.

If $n = 1$ then only (i) applies, and this ordering coincides with the natural one in $\mathbf{Z}_+$. For $n > 1$, if (i) and (ii) are both false, then (iii) means that one must decrease the dimension n by replacing k with its tail $t(k)$, and retry the comparison. For this reason the ordering has been established for $1 \leq m \leq n$. Eventually, one ends up with $m = 1$, to which only (i) applies.

It is convenient to define $\mathcal{P}_n(k)$ as the set of the elements which precede k; formally:

$$\mathcal{P}_n(k) = \{k' \in \mathcal{A}_n \;:\; k' \prec k\}\ .$$

Table 1.1. Ordering of integer vectors in $\mathbf{Z}_+^n$ for $n = 2, 3, 4, 5$.

$I(k)$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	$(0,0)$	$(0,0,0)$	$(0,0,0,0)$	$(0,0,0,0,0)$
1	$(1,0)$	$(1,0,0)$	$(1,0,0,0)$	$(1,0,0,0,0)$
2	$(0,1)$	$(0,1,0)$	$(0,1,0,0)$	$(0,1,0,0,0)$
3	$(2,0)$	$(0,0,1)$	$(0,0,1,0)$	$(0,0,1,0,0)$
4	$(1,1)$	$(2,0,0)$	$(0,0,0,1)$	$(0,0,0,1,0)$
5	$(0,2)$	$(1,1,0)$	$(2,0,0,0)$	$(0,0,0,0,1)$
6	$(3,0)$	$(1,0,1)$	$(1,1,0,0)$	$(2,0,0,0,0)$
7	$(2,1)$	$(0,2,0)$	$(1,0,1,0)$	$(1,1,0,0,0)$
8	$(1,2)$	$(0,1,1)$	$(1,0,0,1)$	$(1,0,1,0,0)$
9	$(0,3)$	$(0,0,2)$	$(0,2,0,0)$	$(1,0,0,1,0)$
10	$(4,0)$	$(3,0,0)$	$(0,1,1,0)$	$(1,0,0,0,1)$
11	$(3,1)$	$(2,1,0)$	$(0,1,0,1)$	$(0,2,0,0,0)$
12	$(2,2)$	$(2,0,1)$	$(0,0,2,0)$	$(0,1,1,0,0)$
13	$(1,3)$	$(1,2,0)$	$(0,0,1,1)$	$(0,1,0,1,0)$
14	$(0,4)$	$(1,1,1)$	$(0,0,0,2)$	$(0,1,0,0,1)$
15	$(5,0)$	$(1,0,2)$	$(3,0,0,0)$	$(0,0,2,0,0)$
16	$(4,1)$	$(0,3,0)$	$(2,1,0,0)$	$(0,0,1,1,0)$
17	$(3,2)$	$(0,2,1)$	$(2,0,1,0)$	$(0,0,1,0,1)$
18	$(2,3)$	$(0,1,2)$	$(2,0,0,1)$	$(0,0,0,2,0)$
19	$(1,4)$	$(0,0,3)$	$(1,2,0,0)$	$(0,0,0,1,1)$
20	$(0,5)$	$(4,0,0)$	$(1,1,1,0)$	$(0,0,0,0,2)$
...	...	...	...	...

With the latter notation the indexing function is simply defined as $I(k) = \#\mathcal{P}_n(k)$. The following definitions are also useful. Pick a vector $k \in \mathcal{A}_m$, and define the sets $\mathcal{B}_m^{(i)}(k)$, $\mathcal{B}_m^{(ii)}(k)$ and $\mathcal{B}_m^{(iii)}(k)$ as the subsets of $\mathcal{A}_m$ satisfying (i), (ii) and (iii), respectively, in the ordering algorithm above. Formally:

$$
\begin{aligned}
\mathcal{B}_m^{(i)}(0) &= \mathcal{B}_m^{(ii)}(0) = \mathcal{B}_m^{(iii)}(0) = \emptyset \ , \\
\mathcal{B}_1^{(ii)}(k) &= \mathcal{B}_1^{(iii)}(k) = \emptyset \ , \\
\mathcal{B}_m^{(i)}(k) &= \{k' \in \mathcal{A}_m \ : \ |k'| < |k|\} \ , \\
\mathcal{B}_m^{(ii)}(k) &= \{k' \in \mathcal{A}_m \ : \ |k'| = |k| \ \wedge \ k'_1 > k_1\} \ , \\
\mathcal{B}_m^{(iii)}(k) &= \{k' \in \mathcal{A}_m \ : \ |k'| = |k| \ \wedge \ k'_1 = k_1 \ \wedge \ t(k') \prec t(k)\} \ .
\end{aligned}
\tag{1.4}
$$

The sets $\mathcal{B}_m^{(i)}(k)$, $\mathcal{B}_m^{(ii)}(k)$ and $\mathcal{B}_m^{(iii)}(k)$ are pairwise disjoint, and moreover

$$\mathcal{B}_m^{(i)}(k) \cup \mathcal{B}_m^{(ii)}(k) \cup \mathcal{B}_m^{(iii)}(k) = \mathcal{P}_m(k) \ .$$

This easily follows from the definition.

1.1.2 Indexing function

Let $k \in \mathcal{A}_n$. In view of the definitions and of the properties above the index function, defined as in (1.1), turns out to be

$$I(0) = 0 \ , \quad I(k) = \#\mathcal{B}_m^{(i)}(k) + \#\mathcal{B}_m^{(ii)}(k) + \#\mathcal{B}_m^{(iii)}(k) \ . \tag{1.5}$$

Let us introduce the functions

$$\begin{aligned} J(n,s) &= \#\{k \in \mathcal{A}_n \ : \ |k| = s\} \ , \\ N(n,s) &= \sum_{j=0}^{s} J(n,j) \qquad\qquad \text{for } n \geq 1 \, , \ s \geq 0 \ . \end{aligned} \tag{1.6}$$

These functions will be referred to in the following as *J–table and N–table.*

I claim that the indexing function can be recursively computed as

$$\begin{aligned} &I(0) = 0 \ , \\ &I(k) = \begin{cases} N(n,|k|-1) & \text{for } k_1 = |k| \ , \\ N(n,|k|-1) + I(t(k)) & \text{for } k_1 < |k| \ , \end{cases} \end{aligned} \tag{1.7}$$

This claim follows from

$$\#\mathcal{B}_m^{(i)}(k) = N(n,|k|-1) \ ; \tag{1.8}$$

$$\#\mathcal{B}_m^{(ii)}(k) = \begin{cases} 0 & \text{for } k_1 = |k| \ , \\ N(n-1,|k|-k_1-1) & \text{for } k_1 < |k| \ ; \end{cases} \tag{1.9}$$

$$\#\mathcal{B}_m^{(iii)}(k) = \begin{cases} I(t(k)) & \text{for } k_1 = |k| \ , \\ I(t(k)) - N(n-1,|k|-k_1-1) & \text{for } k_1 < |k| \ . \end{cases} \tag{1.10}$$

The equality (1.8) is a straightforward consequence of the definition of the N–table. The equality (1.9) follows from (1.4). Indeed, for $|k| = k_1$ we have $\mathcal{B}_m^{(ii)}(k) = \emptyset$, and for $|k| > k_1$ we have

$$\begin{aligned} \mathcal{B}_m^{(ii)}(k) &= \bigcup_{k_1 < j \leq |k|} \{k' \in \mathcal{A}_m \ : \ k'_1 = j \ \wedge \ |t(k')| = |k| - j\} \\ &= \bigcup_{0 \leq l < |k| - k_1} \{k' \in \mathcal{A}_m \ : \ k'_1 = |k| - l \ \wedge \ |t(k')| = l\} \ ; \end{aligned}$$

Coming to (1.10), first remark that

$$\mathcal{B}_m^{(iii)}(k) = \{k' \in \mathcal{A}_m \ : \ k'_1 = k_1 \ \wedge \ |t(k')| = |k| - k_1 \ \wedge \ t(k') \prec t(k)\} \ ,$$

so that

$$\#\mathcal{B}_m^{(iii)}(k) = \#\{\lambda \in \mathcal{A}_{m-1} \,:\, |\lambda| = |k| - k_1 \;\wedge\; \lambda \prec t(k)\} \;.$$

Then, the equality follows by remarking that

$$\mathcal{P}_{m-1}(t(k)) = \{\lambda \in \mathcal{A}_{m-1} \,:\, |\lambda| = |k| - k_1 \;\wedge\; \lambda \prec t(k)\} \cup \{\lambda \in \mathcal{A}_{m-1} \,:\, |\lambda| < |k| - k_1\} \;.$$

Adding up all contributions (1.7) follows.

1.1.3 Construction of the tables

In view of (1.6) and (1.7) the indexing function is completely determined in explicit form by the J–table. I show now how to compute the J–table recursively. For $n = 1$ we have, trivially, $J(1, s) = 1$ for $s \geq 0$. For $n > 1$ use the elementary property

$$\{k \in \mathcal{A}_n \,:\, |k| = s\} = \bigcup_{0\leq j\leq s} \{k \in \mathcal{A}_n \,:\, k_1 = s - j \;\wedge\; |t(k)| = j\} \;.$$

Therefore

$$\begin{aligned} &J(1, s) = 1 \;, \\ &J(n, s) = \sum_{j=0}^{s} J(n-1, j) \;, \quad \text{for } n > 1 \;. \end{aligned} \tag{1.11}$$

This also means that, according to (1.6), we have $N(n, s) = J(n + 1, s)$.

By the way, one will recognize that the J–table is actually the table of binomial coefficients, being $J(n, s) = \binom{n+s-1}{n-1}$.

1.1.4 Inversion of the index function

The problem is to find the vector k of given dimension n corresponding to a given index l.

For $n = 1$ we have $I^{-1}(l) = l$, of course. Therefore, let us assume $n > 1$. I will construct a recursive algorithm which calculates the inverse function by just showing how to determine k_1 and $I\bigl(t(k)\bigr)$.

(i) If $l = 0$, then $k = 0$, and there is nothing else to do.

(ii) If $l > 0$, find an integer s satisfying $N(n, s-1) \leq l < N(n, s)$. In view of (1.7) we have $|k| = s$ and $I(t(k)) = l - N(n, s-1)$. Hence, by the same method, we can determine $\bigl|t(k)\bigr|$, and so also $k_1 = s - \bigl|t(k)\bigr|$.

1.2 Trigonometric polynomials

Let us now consider the more general case $\mathcal{A}_n = \mathbf{Z}^n$. The index n in $\mathcal{A}_n$ denotes again the dimension of the space. The name used in the title of the section is justified because this case occurs precisely in the representation of trigonometric polynomials. Indeed, consider a generic trigonometric expression $f = \sum_{k\in\mathbf{Z}^n} (a_k \cos\langle k, q\rangle + b_k \sin\langle k, q\rangle)$, where $q \in \mathbf{T}^n$ and $a_k\,, b_k$ are numerical coefficients. Recalling that $\cos(-\alpha) = \cos(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$ we can label every term of the trigonometric expression above by a single vector $k \in \mathbf{Z}^n$, provided we agree that if the first nonzero component of k is positive then we take the cos function, else we take the sin function; the case $k = 0$ corresponds to the constant term.

I shall now denote by $|k| = |k_1| + \ldots + |k_n|$ the length (or norm) of the vector $k \in \mathbf{Z}^n$. The tail $t(k)$ of a vector k will be defined as in the previous section.

1.2.1 Ordering relation

Pick a fixed n, and consider the finite family of sets $\mathcal{A}_1 = \mathbf{Z}\,, \ldots, \mathcal{A}_n = \mathbf{Z}^n$.

Let $k, k' \in \mathcal{A}_m$, with any $1 \le m \le n$. I shall say $k' \prec k$ in case one of the following conditions is true:

(i) $m \ge 1 \;\wedge\; |k'| < |k|\,;$
(ii) $m > 1 \;\wedge\; |k'| = |k| \;\wedge\; |k'_1| > |k_1|\,;$
(iii) $m \ge 1 \;\wedge\; |k'| = |k| \;\wedge\; |k'_1| = |k_1| \;\wedge\; k'_1 > k_1$
(iv) $m > 1 \;\wedge\; |k'| = |k| \;\wedge\; k'_1 = k_1 \;\wedge\; t(k') \prec t(k)\,.$

In table 1.2 the order resulting from this definition is illustrated for the cases $n = 2, 3, 4$.

If $n = 1$ this ordering coincides with the ordering in $\mathbf{Z}$ introduced at the beginning of this chapter. For $n > 1$, if (i), (ii) and (iii) do not apply, then (iv) means that one must decrease the dimension n by replacing k with its tail $t(k)$, and retry the comparison. Eventually, one ends up with $m = 1$, falling back to the one dimensional case to which only (i) and (iii) apply.

The ordering in this section has been defined for the case $\mathcal{A}_n = \mathbf{Z}^n$. However, it will be useful to consider particular subsets $\mathbf{Z}^n$. The natural choice will be to use again the ordering relation defined here. For example, the case of integer vectors with non negative components discussed in sect. 1.1.1 can be considered as a particular case: the restriction of the ordering relation to that case gives exactly the order introduced in sect. 1.1.1. Just remark that the condition (iii) above becomes meaningless, so that it can be removed.

The set $\mathcal{P}_n(k)$ of the elements preceding $k \in \mathcal{A}^n$ in the order above is defined as in sect. 1.1.1. Following the line of the discussion in that section it is also convenient to give some more definitions. [1] Pick a vector $k \in \mathcal{A}_n$, and define the sets $\mathcal{B}_m^{(i)}(k)$, $\mathcal{B}_m^{(ii)}(k)$, $\mathcal{B}_m^{(iii)}(k)$ and $\mathcal{B}_m^{(iv)}(k)$ as the subsets of $\mathcal{A}_n$ satisfying (i), (ii), (iii) and (iv),

[1] The sets defined here do not coincide with the corresponding ones in sect. 1.1.1. However, I use the same symbols because they play the same role. In this way I stress that there is a common scheme that works in all cases considered in this chapter. The differences are to be considered as *variazioni* on a given *tema*. By the way, considering several different cases takes more pages that a general synthetic formulation of the most general case, but I hope that the increase of the number of pages will be compensated by a significant decrease of the time spent by the reader in understanding the methods.

Table 1.2. Ordering of integer vectors in $\mathbf{Z}^n$ for $n = 2, 3, 4$.

$I(k)$	$n = 2$	$n = 3$	$n = 4$
0	$(0, 0)$	$(0, 0, 0)$	$(0, 0, 0, 0)$
1	$(1, 0)$	$(1, 0, 0)$	$(1, 0, 0, 0)$
2	$(-1, 0)$	$(-1, 0, 0)$	$(-1, 0, 0, 0)$
3	$(0, 1)$	$(0, 1, 0)$	$(0, 1, 0, 0)$
4	$(0, -1)$	$(0, -1, 0)$	$(0, -1, 0, 0)$
5	$(2, 0)$	$(0, 0, 1)$	$(0, 0, 1, 0)$
6	$(-2, 0)$	$(0, 0, -1)$	$(0, 0, -1, 0)$
7	$(1, 1)$	$(2, 0, 0)$	$(0, 0, 0, 1)$
8	$(1, -1)$	$(-2, 0, 0)$	$(0, 0, 0, -1)$
9	$(-1, 1)$	$(1, 1, 0)$	$(2, 0, 0, 0)$
10	$(-1, -1)$	$(1, -1, 0)$	$(-2, 0, 0, 0)$
11	$(0, 2)$	$(1, 0, 1)$	$(1, 1, 0, 0)$
12	$(0, -2)$	$(1, 0, -1)$	$(1, -1, 0, 0)$
13	$(3, 0)$	$(-1, 1, 0)$	$(1, 0, 1, 0)$
14	$(-3, 0)$	$(-1, -1, 0)$	$(1, 0, -1, 0)$
15	$(2, 1)$	$(-1, 0, 1)$	$(1, 0, 0, 1)$
16	$(2, -1)$	$(-1, 0, -1)$	$(1, 0, 0, -1)$
17	$(-2, 1)$	$(0, 2, 0)$	$(-1, 1, 0, 0)$
18	$(-2, -1)$	$(0, -2, 0)$	$(-1, -1, 0, 0)$
19	$(1, 2)$	$(0, 1, 1)$	$(-1, 0, 1, 0)$
20	$(1, -2)$	$(0, 1, -1)$	$(-1, 0, -1, 0)$
21	$(-1, 2)$	$(0, -1, 1)$	$(-1, 0, 0, 1)$
22	$(-1, -2)$	$(0, -1, -1)$	$(-1, 0, 0, -1)$
23	$(0, 3)$	$(0, 0, 2)$	$(0, 2, 0, 0)$
24	$(0, -3)$	$(0, 0, -2)$	$(0, -2, 0, 0)$
...	...	...	...

respectively, in the ordering algorithm above. Formally,

$$
\begin{aligned}
&\mathcal{B}_m^{(i)}(0) = \mathcal{B}_m^{(ii)}(0) = \mathcal{B}_m^{(iii)}(0) = \mathcal{B}_m^{(iv)}(0) = \emptyset \ ,\\
&\mathcal{B}_1^{(ii)}(k) = \mathcal{B}_1^{(iv)}(k) = \emptyset \ ,\\
&\mathcal{B}_m^{(i)}(k) = \{k' \in \mathcal{A}_m \ : \ |k'| < |k|\} \ ,\\
&\mathcal{B}_m^{(ii)}(k) = \{k' \in \mathcal{A}_m \ : \ |k'| = |k| \ \wedge \ |k'_1| > |k_1|\} \ ,\\
&\mathcal{B}_m^{(iii)}(k) = \{k' \in \mathcal{A}_m \ : \ |k'| = |k| \ \wedge \ |k'_1| = |k_1| \ \wedge \ k'_1 > k_1\} \ ,\\
&\mathcal{B}_m^{(iv)}(k) = \{k' \in \mathcal{A}_m \ : \ |k'| = |k| \ \wedge \ k'_1 = k_1 \ \wedge \ t(k') < t(k)\} \ .
\end{aligned}
\tag{1.12}
$$

The sets $\mathcal{B}_m^{(i)}(k)$, $\mathcal{B}_m^{(ii)}(k)$, $\mathcal{B}_m^{(iii)}(k)$ and $\mathcal{B}_m^{(iv)}(k)$ are pairwise disjoint, and moreover

$$
\mathcal{B}_m^{(i)}(k) \cup \mathcal{B}_m^{(ii)}(k) \cup \mathcal{B}_m^{(iii)}(k) \cup \mathcal{B}_m^{(iv)}(k) = \mathcal{P}(k) \ .
$$

This easily follows from the definition.

1.2.2 Indexing function

Let $k \in \mathcal{A}_n$. In view of the definitions and of the properties above the index function, defined as in (1.1), turns out to be

$$
I(0) = 0 \ , \quad I(k) = \#\mathcal{B}_m^{(i)}(k) + \#\mathcal{B}_m^{(ii)}(k) + \#\mathcal{B}_m^{(iii)}(k) + \#\mathcal{B}_m^{(iv)}(k) \ . \tag{1.13}
$$

Let us introduce the J–table and the N–table as [2]

$$
\begin{aligned}
&J(n,s) = \#\{k \in \mathcal{A}_n \ : \ |k| = s\} \ ,\\
&N(n,s) = \sum_{j=0}^{s} J(n,j) \qquad\qquad \text{for } n \geq 1 \, , \ s \geq 0 \ .
\end{aligned}
\tag{1.14}
$$

I claim that the index function can be recursively computed as

$$
\begin{aligned}
&I(0) = 0 \ ,\\
&I(k) = \begin{cases}
N(n,|k|-1) & \text{for } |k_1| = |k| \ \wedge \ k_1 \geq 0 \ ,\\
N(n,|k|-1) + 1 & \text{for } |k_1| = |k| \ \wedge \ k_1 < 0 \ ,\\
N(n,|k|-1) + N(n-1,|k|-|k_1|-1) + I(t(k)) & \text{for } |k_1| < |k| \ \wedge \ k_1 \geq 0 \ ,\\
N(n,|k|-1) + N(n-1,|k|-|k_1|) + I(t(k)) & \text{for } |k_1| < |k| \ \wedge \ k_1 < 0 \ ,
\end{cases}
\end{aligned}
\tag{1.15}
$$

This formula follows from

$$
\#\mathcal{B}_n^{(i)}(k) = N(n,|k|-1) \ ; \tag{1.16}
$$

$$
\#\mathcal{B}_n^{(ii)}(k) = \begin{cases}
0 & \text{for } |k_1| = |k| \ ,\\
2N(n-1,|k|-|k_1|-1) & \text{for } |k_1| < |k| \ ;
\end{cases}
\tag{1.17}
$$

---

[2] Here too I use the same symbols J and N for the functions, insisting on the fact that they play the same role as in the polynomial case, *not* that they are the same functions. The names J–table and N–table are motivated by the same reason.

$$(1.18)\quad \#\mathcal{B}_n^{(iii)}(k) = \begin{cases} 0 & \text{for } |k_1| \le |k| \ \wedge\ k_1 \ge 0\ , \\ J(n-1, |k| - |k_1|) & \text{for } |k_1| \le |k| \ \wedge\ k_1 < 0\ ; \end{cases}$$

$$(1.19)\quad \#\mathcal{B}_n^{(iv)}(k) = \begin{cases} I(t(k)) & \text{for } |k_1| = |k|\ , \\ I(t(k)) - N(n-1, |k| - |k_1| - 1) & \text{for } |k_1| < |k|\ . \end{cases}$$

The equaliy (1.16) is a straightforward consequence of the definition (1.12). The equality (1.17) follows by remarking that for $|k_1| = |k|$ we have $\mathcal{B}_n^{(ii)}(k) = \emptyset$, and for $|k_1| < |k|$ we have

$$\mathcal{B}_n^{(ii)}(k) = B_n^+(k) \cup B_n^-(k)\ ,\quad B_n^+(k) \cap B_n^-(k) = \emptyset\ ,$$

with

$$B_n^+(k) = \bigcup_{0\le l<|k|-|k_1|} \{k' \in \mathcal{A}_n \ :\ k_1' = |k| - l \ \wedge\ |t(k)| = l\}\ ,$$

$$B_n^-(k) = \bigcup_{0\le l<|k|-|k_1|} \{k' \in \mathcal{A}_n \ :\ k_1' = l - |k| \ \wedge\ |t(k)| = l\}\ ;$$

use also $\#B_n^+(k) = \#B_n^-(k)$. The equality (1.18) follows from

$$\mathcal{B}_n^{(iii)}(k) = \begin{cases} \emptyset & \text{for } |k_1| = |k| \ \wedge\ k_1 \ge 0\ , \\ \{k' \in \mathcal{A}_n \ :\ k_1' = |k_1| \ \wedge\ |t(k')| = |k| - |k_1|\} & \text{for } |k_1| = |k| \ \wedge\ k_1 < 0\ . \end{cases}$$

Coming to (1.19), remark that

$$\mathcal{B}_n^{(iv)}(k) = \{k' \in \mathcal{A}_n \ :\ |t(k')| = |k| - |k_1| \ \wedge\ t(k') \prec t(k)\}\ .$$

Proceeding as in the polynomial case we find again

$$\#\mathcal{B}_n^{(iv)}(k) = \#\{\lambda \in \mathcal{A}_{n-1} \ :\ |\lambda| = |k| - |k_1| \ \wedge\ \lambda \prec t(k)\}\ .$$

and (1.19) follows by remarking that

$$\mathcal{P}_{n-1}(t(k)) = \{\lambda \in \mathcal{A}_{n-1} : |\lambda| = |k| - |k_1| \ \wedge\ \lambda \prec t(k)\} \cup \{\lambda \in \mathcal{A}_{n-1} : |\lambda| < |k| - |k_1|\}\ .$$

Adding up all contributions (1.15) follows.

1.2.3 Construction of the tables

I show now how to construct recursively the J–table, so that the N–table can be constructed, too. For $n = 1$ we have, trivially, $J(1, 0) = 1$ and $J(1, s) = 2$ for $s > 0$. For $n > 1$ use the elementary property

$$\{k \in \mathcal{A}_n \ :\ |k| = s\} = \bigcup_{-s\le j\le s} \{k \in \mathcal{A}_n \ :\ k_1 = j \ \wedge\ |t(k)| = s - |j|\}\ .$$

Therefore

$$(1.20)\quad \begin{aligned} &J(1, 0) = 1\ , \\ &J(1, s) = 2\ , \\ &J(n, s) = \sum_{j=-s}^{s} J(n-1, s-|j|)\ ,\quad \text{for } n > 1\ . \end{aligned}$$

This completely determines the J–table.

1.2.4 Inversion of the index function

The problem is to find the vector k of given dimension n corresponding to the given index l. For $n = 1$ the function $I(k)$ and its inverse $I^{-1}(l)$ are given by (1.2) and (1.3). Therefore in the rest of this section I shall assume $n > 1$. I will give a recursive algorithm, showing how to determine k_1 and $I\bigl(t(k)\bigr)$.

(i) If $l = 0$ then $k = 0$, and there is nothing else to do.

(ii) Assuming that $l > 0$, determine s such that

$$N(n, s-1) \le l < N(n, s) \ .$$

From this we know that $|k| = s$.

(iii) Define $l_1 = l - N(n, s-1)$, so that $I\bigl(t(k)\bigr) \le l_1$ by (1.15). If $l_1 = 0$ set $s_1 = 0$; else, determine s' such that

$$N(n-1, s'-1) \le l_1 < N(n-1, s') \ ;$$

and let $s_1 = \min(s', s)$. In view of $I\bigl(t(k)\bigr) \le l_1$ we know that $|t(k)| \le s_1$. Remark also that $s_1 = 0$ if and only if $l_1 = 0$. For, if $s_1 \ge 1$ then we have $l_1 \ge N(n-1, 0) = 1$.

(iv) If $l_1 = 0$, then by the first of (1.15) we conclude

$$k_1 = |k| = s \ , \quad t(k) = 0 \ ,$$

and there is nothing else to do.

(v) If $l_1 = 1$, then by the second of (1.15) we conclude

$$k_1 = -|k| = -s \ , \quad t(k) = 0 \ ,$$

and there is nothing else to do.

(vi) If $l_1 > 1$ and $s_1 > 0$, we first look if we can set $0 \le k_1 < |k|$. In view of the third of (1.15) we should have

$$|k| - k_1 = s_1 \ , \quad \bigl|t(k)\bigr| = s_1 \ , \quad I\bigl(t(k)\bigr) = l_1 - N(n-1, s_1 - 1) \ .$$

This can be consistently made provided the conditions

$$s_1 > 0 \quad \text{and} \quad I\bigl(t(k)\bigr) \ge N(n-1, s_1 - 1)$$

are fulfilled. The condition $s > 0$ is already satisfied. By (1.15), the second condition is fulfilled provided $l_1 \ge 2N(n-1, s_1-1)$. This has to be checked.

(vi.a) If the second condition is true, then set $k_1 = |k| - s_1$, and recall that $\bigl|t(k)\bigr| = s_1$. Hence, we can replace n, l, and s by $n-1$, $l_1 - N(n-1, s_1-1)$ and s_1, respectively, and proceed by recursion restarting again from the point (iii).

(vi.b) If the second condition is false, then we proceed with the next point.

(vii) Recall that $l_1 > 1$, and remark that we have also $s_1 > 1$. Indeed, we already know $s_1 > 0$, so we have to exclude the case $s_1 = 1$. Let, by contradiction, $s_1 = 1$. Then we have $l_1 \ge 2 = 2N(n-1, s_1-1)$, which is the case already excluded by (vi). We conclude $s_1 > 1$. We look now for the possibility of setting $|k_1| < |k|$ and $k_1 < 0$. In view of the fourth of (1.15) we should have

$$|k| + k_1 = s_1 - 1 \ , \quad \bigl|t(k)\bigr| = s_1 - 1 \ , \quad I\bigl(t(k)\bigr) = l_1 - N(n-1, s_1 - 1) \ .$$

This can be consistently made provided the conditions

$$s_1 > 1 \quad \text{and} \quad I\bigl(t(k)\bigr) \geq N(n-1, s_1 - 2)$$

are fulfilled. The condition $s_1 > 1$ is already satisfied. As to the second condition, by (1.15) it is fulfilled provided $l_1 > N(n-1, s_1-1) + N(n-1, s_1-2)$. This has to be checked.

(vii.a) If the second condition is true, then set $k_1 = -|k| + s_1 - 1$, and recall that $\bigl|t(k)\bigr| = s_1 - 1$. Hence, we can replace n, l, and s by $n-1$, $l_1 - N(n-1, s_1-2)$ and $s_1 - 1$, respectively, and proceed by recursion restarting again from the point (iii).

(vii.b) If the second condition is false we must decrease s_1 by one and start again with the point (vi); remark that $s_1 > 1$ implies $s_1 - 1 > 0$, which is the first of the two conditions to be satisfied at the point (vi), hence the recursion is correct.

Since $l_1 > 1$ we have $l_1 > 2N(n-1, 0)$, so that the conditions of point (vi) are satisfied for $s = 1$. Hence the algorithm above does not fall into an infinite loop between points (vi) and (vii). On the other hand, for $n = 1$ either (iii) or (iv) applies, so that the algorithm stops at that point.