Towards stability results for planetary problems with more than three bodies

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Based on a research work in collaboration with
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The planar Sun–Jupiter–Saturn–Uranus system.

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Answers:
The Kolmogorov's theorem (KAM) was successfully applied to the *realistic* SJS system (L.&G. 2007).
We applied the Nekhoroshev's like exponential estimates for the stability time of the SJS system, in the neighborhood of a KAM torus (G.,L.&S. 2009).
We are studying the secular problem for the Sun–Jupiter–Saturn–Uranus system.

Chaos in the Sun–Jupiter–Saturn–Uranus–Neptune system

Sussman & Wisdom (*Science*, 1992)

- the giant planets subsystem (hereafter SJSUN system) is **chaotic**,
- small changes of the initial conditions can yield quasi–periodic motions.

Murray & Holman (*Science*, 1999) *overlap of some three–body resonances*

- the actual value of the Uranus semi–major axis is located very close to the center of the cluster of the following type of resonances:

  $$3n_J - 5n_S - 7n_U + [(3-j)g_J + 6g_S + jg_U] \qquad \text{with} \ \ j = 0,\,1,\,2,\,3\ ,$$

  where n stands for the mean motion frequency of a planet and g means the (secular) frequency of its perihelion argument,
- other cluster of resonances "generated" by the $(3,-5,-7)$ mean motion resonance; some of them include also the frequencies related to the longitude of the nodes.

Dynamics of the SJSUN and SJSU systems

The triple $(3, -5, -7)$ mean motion resonance is so relevant, because Saturn is close to the celebrated $5 : 2$ resonance with Jupiter, while Uranus is near to the $7 : 1$. Moreover, $2n_J - 5n_S \simeq 7n_U - n_J$.

M.&H. roughly evaluated the time T_{ej} needed by Uranus to be ejected as $T_{ej} \sim 10^{18}$.

Result (given by numerical explorations):

When a_U (Uranus semi–major axis) ranges between 19.18 and 19.35 AU, some regions looks filled by quasi–periodic motions (i.e. with Lyap. time $> 10^8$), while some other regions are chaotic due to the effect of clusters of three–body resonances.

This result *(qualitatively) still holds true* either *in the planar case* or *after having removed Neptune*.
On the other hand, in that whole region M.&H. did not detect chaotic motions *in the planar case without Neptune*.

Normal forms approach to the SJSU and SJSUN systems

Remark:

From the study of the Sun–Jupiter–Saturn system we know that a careful handling of *the secular part of the Hamiltonian is crucial*. Before starting to manipulate the secular part of the Hamiltonian, we need to reduce the main part of the perturbation depending on the fast angles.

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Strategy options (after having added, at least, Uranus):

It looks natural to use stability times estimates about some Birkhoff normal forms (roughly, as for the “exponential estimates” leading to Nekhorohev's like results). It could be done in one of the following ways:

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3. nearby a KAM torus (as in G.,L.&S., *Cel. Mech. Dyn. Astr.*, 2009).

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1. by averaging with respect to the mean motion angles and by eliminating the secular perturbing terms corresponding to orders higher than 2 both in eccentricities and inclinations;
2. in the neighborhood of an elliptic torus;
3. nearby a KAM torus (as in G.,L.&S., *Cel. Mech. Dyn. Astr.*, 2009).

Application to the planar SJSU system

- The **planar** Sun–Jupiter–Saturn–Uranus (shortly, SJSU) system.
- Huge amount of calculations in the “Kolmogorov’s like” step.
- **No** reduction of the angular momentum.
- Expansion of the Hamiltonian using an algebraic manipulator.
- Study of the secular dynamics.
- Estimate of the “stability time”.

The Hamiltonian of the planetary system

The Hamiltonian is

$$F(\mathbf{r},\tilde{\mathbf{r}}) = T^{(0)}(\tilde{\mathbf{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\tilde{\mathbf{r}}) + U^{(1)}(\mathbf{r})\,,$$

where $\mathbf{r}$ are the heliocentric coordinates and $\tilde{\mathbf{r}}$ the conjugated momenta.

$$T^{(0)}(\tilde{\mathbf{r}}) = \frac{1}{2}\sum_{j=1}^{3}\|\tilde{\mathbf{r}}_j\|^2\left(\frac{1}{m_0}+\frac{1}{m_j}\right)\ ,$$

$$U^{(0)}(\mathbf{r}) = -\mathcal{G}\sum_{j=1}^{3}\frac{m_0\,m_j}{\|\mathbf{r}_j\|}\ ,$$

$$T^{(1)}(\tilde{\mathbf{r}}) = \frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_2}{m_0}+\frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_3}{m_0}+\frac{\tilde{\mathbf{r}}_2\cdot\tilde{\mathbf{r}}_3}{m_0}\ ,$$

$$U^{(1)}(\mathbf{r}) = -\mathcal{G}\left(\frac{m_1\,m_2}{\|\mathbf{r}_1-\mathbf{r}_2\|}+\frac{m_1\,m_3}{\|\mathbf{r}_1-\mathbf{r}_3\|}+\frac{m_2\,m_3}{\|\mathbf{r}_2-\mathbf{r}_3\|}\right)\ .$$

The Poincaré variables in the plane

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j$$

$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j) \quad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j)$$

where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and perihelion argument of the j–th planet, respectively.

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The Hamiltonian in the Poincaré variables

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$$F = F_0 + F_1 = F_0 + U^{(1)} + T^{(1)}$$

$$F_0 = -\sum_{i=1}^{n} \frac{\mu_i^2 \beta_i^3}{2\Lambda_i^2} \qquad \text{integrable part}\,,$$

$$U^{(1)} = -G \sum_{0<i<j} \frac{m_i m_j}{\Delta_{ij}} \qquad \text{perturbation (main term)}\,,$$

$$T^{(1)} = \sum_{0<i<j} \frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0} \qquad \text{perturbation (complementary term)}\,.$$

We need to expand all this terms in power series!

How to expand the Hamiltonian

1. The development of the Hamiltonian is a quite standard matter.
2. Choose a Λ^* such that

$$\left.\frac{\partial\langle F\rangle_\lambda}{\partial\Lambda_j}\right|_{\substack{\Lambda=\Lambda^*\\ \xi=\eta=0}} = n_j^*\,, \qquad j=1,2,3\,.$$

   - $\langle.\rangle_\lambda$ means the average over the fast angles ,
   - n_j^* are the fundamental frequencies of the mean motion .
3. Introduce new actions $L_j = \Lambda_j - \Lambda_j^*$.
4. Perform the canonical transformation $\mathcal{T}_F$ translating the fast actions.
5. Expand the Hamiltonian in power series of $\mathbf{L}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and in Fourier series of $\boldsymbol{\lambda}$.

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The expansion of the Hamiltonian

For what concerns the classical expansions of the Hamiltonian in canonical variables we essentially follow

- Laskar & Robutel, *CeMDA*, 1995,
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$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where $h_{j_1,0}^{(Kep)}$ is an homogeneous polynomial of degree j_1 in $\mathbf{L}$ and

$$h_{j_1,j_2}^{(\mathcal{T}_F)} \text{is a } \begin{cases} \text{hom. pol. of degree } j_1 \text{ in } \mathbf{L}\,, \\ \text{hom. pol. of degree } j_2 \text{ in } \boldsymbol{\xi}, \boldsymbol{\eta}\,, \\ \text{with coeff. that are trig. pol. in } \boldsymbol{\lambda}\,. \end{cases}$$

Truncation limits of the expansion

This is the Hamiltonian,

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

Truncation limits of the expansion

This is the computed Hamiltonian,

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{2} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{1} \sum_{j_2=0}^{12} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where we also truncate all the coefficients with harmonics of degree greater than 16.

These are the lowest limits to include the fundamental features of the system.

The scheme of the preliminary perturbation reduction

Our goal is to try to eliminate the terms

$$\left\lceil \mu\, h_{0,0}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}),\ \left\lceil \mu\, h_{0,1}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}),\ \ldots,\ \left\lceil \mu\, h_{0,6}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}),$$

and

$$\left\lceil \mu\, h_{1,0}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}),\ \left\lceil \mu\, h_{1,1}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}),\ \ldots,\ \left\lceil \mu\, h_{1,6}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}).$$

where $\lceil\cdot\rceil_{\boldsymbol{\lambda}:K}$ means the truncation of the harmonics of degree greater than K.

The details of the transformation

This procedure is essentially a "Kolmogorov's like" step of normalization.

If you want to kill the term $\left\lceil h_{j_1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:K}$, than you have to solve the homological equation

$$\{\chi,\, \mathbf{n}^* \cdot \mathbf{L}\} + \left\lceil \mu\, h_{j_1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:K} = 0\,,$$

where

$$\{\chi,\, f\} = (\partial_{\mathbf{L}}\chi \cdot \partial_{\boldsymbol{\lambda}} f - \partial_{\boldsymbol{\lambda}}\chi \cdot \partial_{\mathbf{L}} f) + (\partial_{\boldsymbol{\xi}}\chi \cdot \partial_{\boldsymbol{\eta}} f - \partial_{\boldsymbol{\eta}}\chi \cdot \partial_{\boldsymbol{\xi}} f)\ ,$$

and find the generating function χ.

The generating function χ has the same structure of $h_{j_1,j_2}^{(\mathcal{T}_F)}$, is of order $\mathcal{O}(\mu)$ and must depends on the fast angles $\boldsymbol{\lambda}$.

The details of the transformation

Using the Lie series formalism, we perform a canonical transformation of the Hamiltonian

$$\exp \mathcal{L}_{\chi} H = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}_{\chi}^{j} H \,,$$

where $\mathcal{L}_{\chi} H = \{\chi,\, H\}$.

This transformation, by construction, kill the term $\left\lceil \mu\, h_{j_1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:K}$, but the transformed Hamiltonian still has a term of the same type, but at least of order $\mathcal{O}(\mu^2)$.

This effect is due to Lie series algorithm, for example take $j_1 = 0$, $j_2 = 0$ and consider the Poisson bracket

$$\left\{ \chi \,,\, \mu\, h_{1,0}^{(\mathcal{T}_F)} \right\} \to \mu^2\, \widetilde{h}_{0,0}^{(\mathcal{T}_F)} \,.$$

Partial preliminary reduction of the perturbation

First step

$$\begin{cases} \mathbf{n}^* \cdot \dfrac{\partial \chi_1^{(\mathcal{O}2)}}{\partial \boldsymbol{\lambda}} + \mu \displaystyle\sum_{j_2=0}^{6} \left\lceil h_{0,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \\ \widetilde{H} = \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} H = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}^j_{\chi_1^{(\mathcal{O}2)}} H \,. \end{cases}$$

Second step

$$\begin{cases} \mathbf{n}^* \cdot \dfrac{\partial \chi_2^{(\mathcal{O}2)}}{\partial \boldsymbol{\lambda}} + \mu \displaystyle\sum_{j_2=0}^{6} \left\lceil \tilde{h}_{1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \\ H^{(\mathcal{O}2)} = \exp \mathcal{L}_{\chi_2^{(\mathcal{O}2)}} \circ \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} H \,. \end{cases}$$

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Why these limits?

The secular variables:

$$\widetilde{H} \qquad\qquad \chi_2^{(\mathcal{O}2)} \qquad \longrightarrow \qquad H^{(\mathcal{O}2)}$$

The fast angles:

$(3, -5, -7)$ harmonics of order 15

Why these limits?

The secular variables:

$$\begin{array}{ccccc} \widetilde{H} & \chi_2^{(\mathcal{O}2)} & \longrightarrow & H^{(\mathcal{O}2)} \\ \downarrow & \downarrow & & \downarrow \\ \cos(1\lambda_1 - 7\lambda_3) & \sin(1\lambda_1 - 7\lambda_3) & & \text{secular terms} \end{array}$$

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The fast angles:

$$(3, -5, -7) \text{ harmonics of order } 15 < \boxed{16}.$$

The Hamiltonian *up to order two in the masses*

- $H^{(\mathcal{O}2)}$ is the Hamiltonian up to order two in the masses.
- **No** terms corresponding to any *triple resonances* **before** the "Kolmogorov's like" step.
- The "Kolmogorov's like" step introduce the *triple resonances*, in particular the $(3, -5, -7)$ resonance.
- Small limits **don't** mean small expansion!
- After the "Kolmogorov's like" step, we have 94 109 751 coefficients.

The secular part *up to order two in the masses*

- Reduction to the secular system:
  - average over the fast angles $\boldsymbol{\lambda}$, and put $\mathbf{L}=0$;
  - hereafter, we are considering a system with *three degrees of freedom*.
- From the D'Alembert rules, it follows that

$$H^{(sec)} = H_0 + H_2 + H_4 + \ldots \ ,$$

where H_{2j} is a hom. pol. of degree $(2j+2)$ in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, $\forall\ j\in\mathbf{N}$.

- $\boldsymbol{\xi}=\boldsymbol{\eta}=0$ is an elliptic equilibrium point.
- We diagonalize the quadratic term by a linear canonical transformation $\mathcal{D}$:

$$H_2^{(\mathcal{D})} = \sum_{j=1}^{3} \frac{\nu_j}{2}\left(\xi_j^2+\eta_j^2\right) \ .$$

- Hereafter, we simply denote with H the secular Hamiltonian having the quadratic part in diagonal form.

Birkhoff normalization of the secular Hamiltonian

- Consider the secular Hamiltonian having the quadratic part in diagonal form:

$$H = H_0 + H_2 + H_4 + \ldots .$$

- Focus on the actions $\Phi_j = \frac{1}{2}\left(\xi_j^2 + \eta_j^2\right)$ with $j = 1, 2, 3$.

- Perform the Birkhoff normalization up to order N:

$$H^{(N)} = Z_0^{(N)} + Z_2^{(N)} + \ldots + Z_N^{(N)} + R_{N+1}^{(N)} + \ldots ,$$

where $Z_0^{(N)}$, $Z_2^{(N)}$, ..., $Z_N^{(N)}$ just depend on Φ_1, Φ_2, Φ_3.

- The time derivative

$$\dot{\Phi}_j = \{\Phi_j, H\} = \sum_{j>N} \left\{\Phi_j, R_j^{(N)}\right\} \simeq \left\{\Phi_j, R_{N+1}^{(N)}\right\}$$

Study of the stability of the secular Hamiltonian

We have

$$\|\Phi(t)-\Phi(0)\| \le \left|\sup_{(\boldsymbol{\xi},\boldsymbol{\eta})\in\Delta_{\rho\mathbf{R}}} \dot{\Phi}(\boldsymbol{\xi},\boldsymbol{\eta})\right| \, |t| \ ,$$

where $\Delta_{\rho\mathbf{R}} = \left\{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{R}^6 : \ \xi_j^2+\eta_j^2 \le \rho^2 R_j^2, \ j=1,2,3\right\}$.

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We want

$$\|\Phi(t)-\Phi(0)\| < \rho - \rho_0 \qquad \forall \ |t| < T \ ,$$

where T is a (eventually long) “stability time”.

Study of the stability of the secular Hamiltonian

We have

$$\|\Phi(t)-\Phi(0)\| \le \left| \sup_{(\boldsymbol{\xi},\boldsymbol{\eta})\in\Delta_{\rho\mathbf{R}}} \dot{\Phi}(\boldsymbol{\xi},\boldsymbol{\eta}) \right| \, |t| \ ,$$

where $\Delta_{\rho\mathbf{R}} = \left\{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{R}^6 : \ \xi_j^2 + \eta_j^2 \le \rho^2 R_j^2, \ j = 1,2,3\right\}$.

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define the norm

$$\|f\|_{\mathbf{R}} = \sum_{|\mathbf{j}+\mathbf{k}|=s} |f_{\mathbf{j},\mathbf{k}}|\mathbf{R}^{\mathbf{j}+\mathbf{k}}$$

The "stability time"

$$T(\rho_0, \rho, N) \lesssim \frac{\rho - \rho_0}{\left\| \left\{ \Phi_j ,\ R_{N+1}^{(N)} \right\} \right\|_{\mathbf{R}} \rho^{N+3}} \ ,$$

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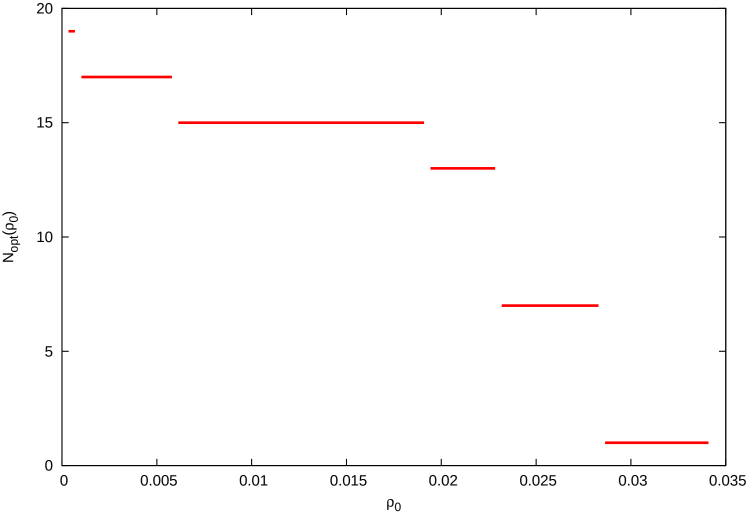
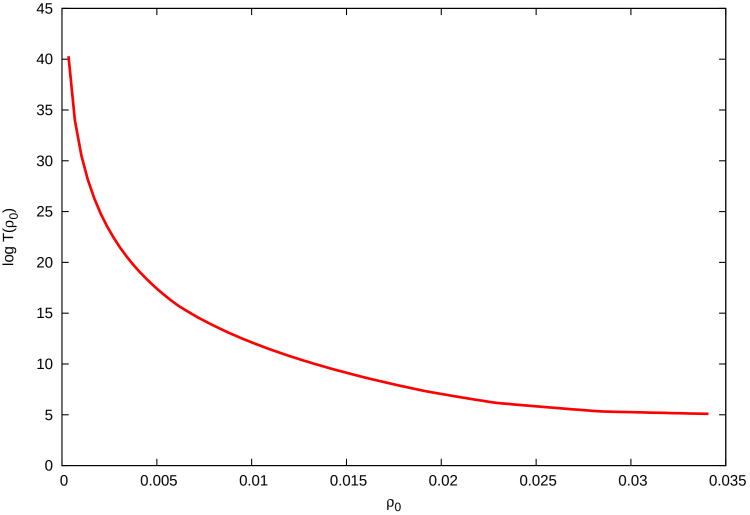
- The "optimal stability time" $T\big(\rho_0, \rho_{opt}(\rho_0, N_{opt}(\rho_0)), N_{opt}(\rho_0)\big)$ depends only on the initial radius ρ_0.

The optimal normalization order



The estimated “stability time” of the secular Hamiltonian



Comments about our results

- We considered a secular Hamiltonian model of the **planar Sun–Jupiter–Saturn–Uranus system**, providing an approximation of the motions of the secular variables *up to order two in the masses*. Our results ensure that such a system is *stable for a time comparable to the age of the universe* just in a domain with a radius that is about **a half of the real distance of the initial secular variables from the origin**.
- We are confident that the reduction of the angular momentum before the "Kolmogorov's like" step will improve significantly our results and this is what we plan to do in a near future.
- If the reduction of the angular momentum won't improve our estimates, we have to introduce some new ideas.

...end

Thanks for your attention.

Questions?

Comments?