Explicit Construction of Elliptic Tori for Planetary Systems

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The Classical KAM Theorem (non–degenerate)

Since the birth of the KAM theory (see [8], [10] and [1]), the invariant tori are expected to be the fundamental dynamical objects which explain the quasi–periodicity of the planetary motions of our Solar System; but the original statements of the KAM theorem apply to quasi–integrable *non–degenerate* Hamiltonian system.

The Planetary KAM Theorem (properly degenerate)

Arnold extended his version of the KAM theorem to *properly degenerate* systems subject to small perturbations (see [2]). About forty years later, it has been shown that this Arnold's theorem can be applied to spatial planetary three–body problems where masses, eccentricities and inclinations are (unrealistically) small enough (see [12]).

Under suitable hypotheses, this Arnold's theorem claims that there is a Cantor set of positive measure formed by the union of invariant tori.

Lower Dimensional Tori (non–constructive approach)

Among the consequences of the statements concerning the invariant tori of maximal dimension, one expects that the persistence under small perturbations should hold also for the n–dimensional invariant tori related to the limit case of small circular orbits of the secular motions (describing, e.g. the behaviour of both the eccentricities and the inclinations). However, a different proof is needed in order to ensure the existence of these lower dimensional tori which are said to be elliptic. Such a theorem has been recently proved by Biasco, Chierchia and Valdinoci in two different cases: for the spatial three–body planetary problem and for a planar system with a central star and n planets (see [4] and [5], respectively). In our opinion, **their approach** is deep from a theoretical point of view, but **is not suitable for explicit applications**, even if one is interested just in finding the location of the elliptic invariant tori.

They apply a theorem due to Pöschel (see [11]), to ensure the existence of the elliptic tori. Pöschel's version of this theorem is based on a careful adaptation of the usual Arnold's proof scheme for non–degenerate systems: the perturbation is removed by a sequence of canonical transformations which are defined on a subset of the phase space excluding the "resonant regions" (see [1]). Since **the resonances are dense everywhere**, the change of coordinates giving the shape of the invariant elliptic tori is defined on a Cantor set which *does not contain any open subset*. The efficiency of an eventual explicit application based of such an approach is highly questionable and, as far as we know, *it has never been used to calculate an orbit of a Celestial Mechanics problem.*

The Kolmogorov's Scheme (constructive approach)

The original proof scheme of the KAM theorem introduced by Kolmogorov is in a much better position for what concerns the translation in an **explicit algorithm** for the construction of invariant tori (see [8], [3] and [9]). In the present work we want to adapt the Kolmogorov's algorithm, in order to construct a suitable normal form related to the elliptic tori. Moreover, this will allow us to explicitly integrate the motions on those invariant surfaces, by using a so called semi–analytic procedure.

Let us emphasize that here we focus just on a direct application to a planetary system. Thus, we will check the effectiveness of our semi–analytic procedure, by calculating a finite number of steps of the algorithm by algebraic manipulations on a calculator. On the other hand, the theoretical study of the convergence of our algorithm is not investigated here and it is deferred to a future work, where we plan to translate our approach in a rigorous proof ensuring the existence of the elliptic tori.

Why is it Relevant to Study Elliptic Invariant Tori?

When one is interested in showing the long term stability of a planetary system, the construction of a normal form related to some fixed elliptic torus could be a relevant milestone. In fact, it is possible to ensure the effective stability in the neighborhood of such an invariant surface by implementing a partial construction of the Birkhoff normal form (see, e.g., [7] and [6], where this approach is used in order to study the stability nearby an invariant KAM torus having maximal dimension).

The location of the elliptic tori can be useful also for practical purposes. In fact, the regions close to them are exceptionally stable, being mainly filled by invariant tori of maximal dimension. Therefore, they can be of interest for spatial missions aiming, for instance, to observe asteroids not far from the elliptic tori. Moreover, our technique should adapt quite easily also to the construction of hyperbolic tori that can be used in the design of spacecraft missions with transfers requiring low energy. Also in view of this kind of applications, lower dimensional tori of elliptic, hyperbolic and mixed type have been studied in the vicinity of the Lagrangian points for both the restricted three–body problem and the bicircular restricted four–body problem (see the works by Gabern, Jorba, Villanueva and others).

Classical expansion of the planar planetary Hamiltonian

In order to apply our methods to a model similar to our outer Solar System, we study an approximation of the planar Sun–Jupiter–Saturn–Uranus system. Let us consider four point bodies with masses m_0, m_1, m_2, m_3, mutually interacting according to Newton's gravitational law (the indexes 0, 1, 2, 3 correspond to Sun, Jupiter, Saturn and Uranus).

The set of the planar Poincaré's canonical variables can be introduced as

$$\Lambda_j = \frac{m_0\, m_j}{m_0 + m_j}\sqrt{G(m_0+m_j)a_j}\ , \qquad \lambda_j = M_j + \omega_j\ ,$$
$$\xi_j = \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\,\cos\omega_j\ , \qquad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\,\sin\omega_j\ ,$$

for $j = 1\,,\,2\,,\,3$, where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and the perihelion argument, respectively, of the j–th planet.

We proceed now by expanding the Hamiltonian in order to construct the first basic approximation of the normal form for elliptic tori. After having chosen a center value Λ^*, we perform a translation defined as $L_j = \Lambda_j - \Lambda_j^*$, for $j = 1, 2, 3$.

The translated Hamiltonian can be expanded in power series of L, ξ, η around the origin,

$$H(L,\lambda,\xi,\eta) = n^*\cdot L + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(L) + \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} f_{j_1,j_2}(L,\lambda,\xi,\eta)\ ,$$

where n^* is vector of the fast frequencies related to the chosen Λ^* and the functions f_{j_1,j_2} are homogeneous polynomials of degree j_1 in the actions L and of degree j_2 in the secular variables (ξ,η). The coefficients of such homogeneous polynomials do depend analytically and periodically on the angles λ. The terms $h_{j_1,0}^{(\mathrm{Kep})}$ of the Keplerian part are homogeneous polynomials of degree j_1 in the actions L.

The equation of motion

Consider any point of the type $(0,\lambda,0,0) \in (0,\mathbf{T}^3,0,0)$, we have

$$\dot L_j = -\frac{\partial f_{0,0}}{\partial\lambda_j}(0,\lambda,0,0) \qquad \dot\xi_j = -\frac{\partial f_{0,1}}{\partial\eta_j}(0,\lambda,0,0)$$
$$\dot\lambda_j = n^* + \frac{\partial f_{1,0}}{\partial L_j}(0,\lambda,0,0) \qquad \dot\eta_j = \frac{\partial f_{0,1}}{\partial\xi_j}(0,\lambda,0,0)$$

so, to make the manifold $(0,\mathbf{T}^3,0,0)$ invariant, we need to kill the terms $f_{0,0}$, $f_{0,1}$ and $f_{1,0}$.

The Normalization Procedure

In order to obtain the normal form related to the elliptic tori of the Hamiltonian at order r, say $H^{(r)}$, starting from $H^{(0)} = H$, we perform r normalization steps using a composition of r Lie series to do the canonical transformation:

$$\mathcal{C}^r = \exp\left(\mathcal{L}_{\chi_2^r}\right)\circ\exp\left(\mathcal{L}_{\chi_1^r}\right)\circ\exp\left(\mathcal{L}_{\chi_0^r}\right)\circ\ldots\circ\exp\left(\mathcal{L}_{\chi_2^1}\right)\circ\exp\left(\mathcal{L}_{\chi_1^1}\right)\circ\exp\left(\mathcal{L}_{\chi_0^1}\right)$$

where the generating functions χ_0^r, χ_1^r and χ_2^r have trigonometric degree smaller than $2r$ and are of degree j_1 in the actions L and j_2 in the secular variables (ξ,η), with $2j_1 + j_2 = 0,\ 1,\ 2$, respectively.

These generating functions are determined so that the limit canonical transformation $\mathcal{C}^\infty$ removes all the unwanted terms, i.e. $f_{0,0}$, $f_{0,1}$ and $f_{1,0}$, from the starting Hamiltonian.

By solving step-by-step the homological equations for the generating functions, as usual, the small divisors arise and we must assume the **first** and the **second Melnikov** conditions. Let us recall that when we implement such an algorithm to a calculator, we can perform just a finite number of steps, therefore the Melnikov conditions need to be fulfilled **only up to a finite order**.

Application to the Planar Sun–Jupiter–Saturn–Uranus System

We consider a planar SJSU model adopting the following truncations criteria:

(a) the Keplerian part is expanded up to the quartic terms;

(b) the terms f_{j_1,j_2} include:

- (b1) the terms having degree j_1 in the actions L with $j_1 \le 3$;
- (b2) all terms having degree j_2 in the secular variables (ξ,η), with j_2 such that $2j_1 + j_2 \le 8$;
- (b3) all terms up to the trigonometric degree 18 with respect to the angles λ.

Let us denote by $\mathcal{E}$ the function

$$\mathcal{E}(L,\lambda,\xi,\eta) = (x,y,p_x,p_y)\,,$$

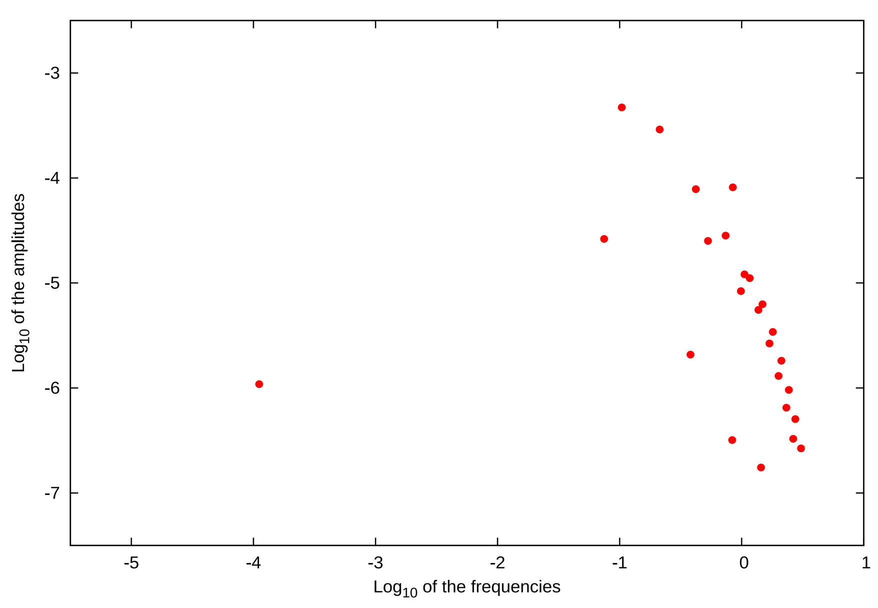
that transform the point (L,λ,ξ,η) into the corresponding position-momenta coordinates.

The sets of initial conditions corresponding to the elliptic torus is given by $\mathcal{E}\circ\mathcal{C}^\infty(0,\lambda,0,0)$, with $\lambda\in\mathbf{T}^3$. By using computer algebra we have been able to construct the approximation of the elliptic torus corresponding to $\mathcal{E}\circ\mathcal{C}^9$.

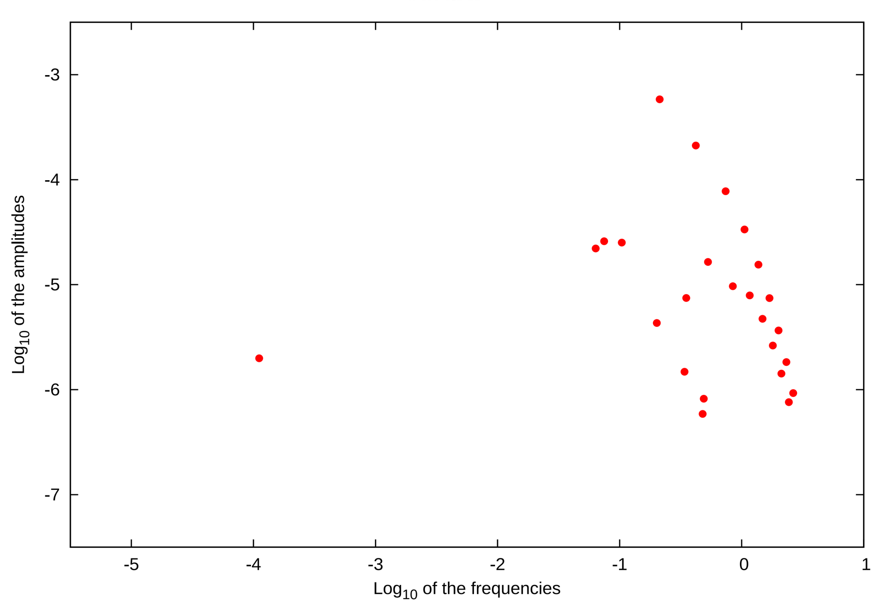
The Fourier spectrum of the motions on elliptic tori is strongly characteristic, in fact **just the fast frequencies and their linear combinations can show up**. This simple remark allows us to check the accuracy of our results by using frequency analysis.

Starting from the integration of the initial condition $\mathcal{E}\circ\mathcal{C}^9(0,0,0,0)$, we perform the frequency analysis of the three signals $\big(\xi_j(t),\eta_j(t)\big)$ for $j = 1\,,\,2\,,\,3$. In the figures below we plot the amplitude versus the frequencies (in a Log-Log scale) for all the first 25 components of the signals.

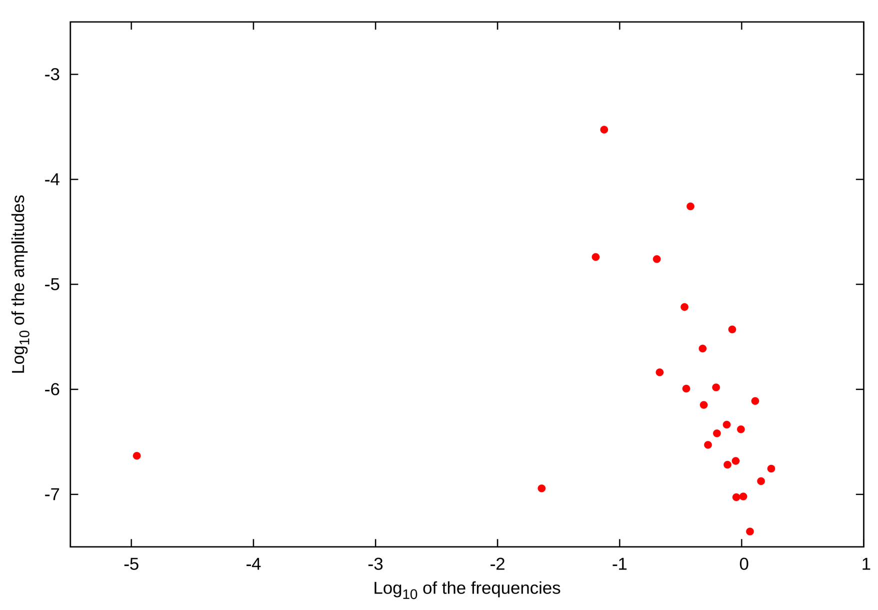
Jupiter



Saturn



Uranus



Conclusions

We can say that secular frequencies have values not greater than 10^{-3}. From the figures above we can remark that nearly all the components correspond to linear combinations of fast frequencies. Each plot highlights the occurrence of just one component corresponding to a secular frequency (with an amplitude that is much smaller than the main ones). This shows that our algorithm is effective.

However, let us recall that the occurrence of secular components are unavoidable in a practical application, due to truncations and numerical errors.

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