On the Secular Evolution of Extrasolar Planetary Systems

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**1995:** first detection of an **extrasolar planet**, a planet orbiting around a star other than the Sun.

Nowadays **more than 700** have been detected, and more than **2000** candidates.
The main difference between the extrasolar systems and the Solar System regards the shape of the orbits.

In the extrasolar systems, the majority of the orbits describe true ellipses (high eccentricities) and no more almost circular like in the Solar System.

The classical approach uses the circular approximation as a reference. Dealing with systems with high eccentricities we need to compute the expansion at high order to study the long-term evolution of the extrasolar planetary system.
Aim of the work

Questions:

- Can we predict the long-term evolution of the extrasolar systems?
- How to identify systems that are near to a mean motion resonance?
- Are the discovered extrasolar systems stable?
- What is the meaning of the word stable?
Aim of the work

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- What is the meaning of the word stable?

Effective stability:

a) Will the orbits of the planets remain essentially the same for eternity, or at least for a time comparable to the estimated age of the Universe?

b) In the distant future, could dramatic changes happen? For example due to collisions, falls on the star or ejection of a planet.
Hamiltonian of the Three-Body Problem

\[ F(\mathbf{r}, \mathbf{\tilde{r}}) = T^{(0)}(\mathbf{\tilde{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\mathbf{\tilde{r}}) + U^{(1)}(\mathbf{r}), \]

where \( \mathbf{r} \) are the heliocentric coordinates and \( \mathbf{\tilde{r}} \) the conjugated momenta.

\[ T^{(0)}(\mathbf{\tilde{r}}) = \frac{1}{2} \sum_{j=1}^{2} \| \mathbf{\tilde{r}}_j \|^2 \left( \frac{1}{m_0} + \frac{1}{m_j} \right), \]

\[ U^{(0)}(\mathbf{r}) = -G \sum_{j=1}^{2} \frac{m_0 m_j}{\| \mathbf{r}_j \|}, \]

\[ T^{(1)}(\mathbf{\tilde{r}}) = \frac{\mathbf{\tilde{r}}_1 \cdot \mathbf{\tilde{r}}_2}{m_0}, \]

\[ U^{(1)}(\mathbf{r}) = -G \frac{m_1 m_2}{\| \mathbf{r}_1 - \mathbf{r}_2 \|}. \]
Orbital Parameters

\( a \) (semi-major axis), \( \omega \) (periapsis argument),
\( e \) (eccentricity), \( M \) (mean anomaly),
\( i \) (inclination), \( \Omega \) (longitude of the ascending node).
Poincaré variables in the plane

\[ \Lambda_j = \frac{m_0 m_j}{m_0 + m_j} \sqrt{G(m_0 + m_j) a_j}, \quad \lambda_j = M_j + \omega_j, \]

\[ \xi_j = \sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2 \cos(\omega_j)}}, \quad \eta_j = -\sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2 \sin(\omega_j)}}, \]

where \( a_j, e_j, M_j \) and \( \omega_j \) are the semi-major axis, the eccentricity, the mean anomaly and periapsis argument of the \( j \)-th planet, respectively.
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Fast variables, \( \mathcal{O}(1 \text{ year}) \)

\[ \xi_j = \sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2 \cos(\omega_j)}}, \quad \eta_j = -\sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2 \sin(\omega_j)}}, \]

Secular variables, \( \mathcal{O}(10000 \text{ years}) \)

where \( a_j, e_j, M_j \) and \( \omega_j \) are the semi-major axis, the eccentricity, the mean anomaly and periapsis argument of the \( j \)-th planet, respectively.
In order to compute the Taylor expansion of the Hamiltonian around the fixed value $\Lambda^*$, we have to introduce the *translated fast action*

$$L = \Lambda - \Lambda^*.$$ 

This is the Hamiltonian,

$$H^{(\mathcal{T})} = n^* \cdot L + \sum_{j_1=2}^{\infty} h^{(\text{Kep})}_{j_1,0}(L) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h^{(\mathcal{T})}_{j_1,j_2}(L, \lambda, \xi, \eta).$$
Expansion of the Hamiltonian

In order to compute the Taylor expansion of the Hamiltonian around the fixed value $\Lambda^*$, we have to introduce the *translated fast action*

$$L = \Lambda - \Lambda^*.$$ 

This is the **computed** Hamiltonian,

$$H^{(T)} = n^* \cdot L + \sum_{j_1=2}^{2} h_{j_1,0}^{(Kep)}(L) + \mu \sum_{j_1=0}^{1} \sum_{j_2=0}^{SEC} h_{j_1,j_2}^{(T)}(L, \lambda, \xi, \eta),$$

where we also truncate all the coefficients with harmonics of degree greater than $2K$.

We will choose the **lowest possible limits** in order to include the **fundamental features** of the system.
In order to **get rid of the fast motion**, we consider the averaged Hamiltonian

\[ H(\mathbf{L}, \lambda, \xi, \eta) = \langle H(\mathbf{L}, \lambda, \xi, \eta) \rangle_\lambda. \]

This is the so called **first order averaging**.

Actually it means that we **remove** from the Hamiltonian all the terms that depends on the fast angles \( \lambda \).
Secular Dynamics

- Doing the averaging over the fast angles (as we are interested in the secular motions of the planets), the system pass from 4 to 2 degrees of freedom,

\[ H^{(\text{sec})} = H_0(\xi, \eta) + H_2(\xi, \eta) + H_4(\xi, \eta) + \ldots, \]

where \( H_{2j} \) is a hom. pol. of degree \((2j + 2)\) in \((\xi, \eta)\), \(\forall j \in \mathbb{N}\).

- \( \xi = \eta = 0 \) is an elliptic equilibrium point, so we can introduce action-angle variables via Birkhoff normal form.

- In this way we remove the dependency on the angles from the Hamiltonian and we can easily solve the equations of motion.
Sketch of the procedure — (1/2)

\[ H = H_0(\xi, \eta) + H_2(\xi, \eta) + H_4(\xi, \eta) + \ldots . \]

- Introduce the **action-angle variables**

  \[ \xi_j = \sqrt{2\Phi_j} \cos \varphi_j , \]

  \[ \eta_j = \sqrt{2\Phi_j} \sin \varphi_j . \]

- Compute the **Birkhoff normal form** up to a finite order, \( r \),

\[ H^{(r)} = Z_0(\Phi) + Z_2(\Phi) + \ldots + Z_r(\Phi) + R_r(\Phi, \varphi) . \]
If the remainder $R_r$ is *small enough*, we can neglect it!

- **The equations of the motion** are

  $$\dot{\Phi}_j(0) = 0, \quad \dot{\varphi}_j(0) = \frac{\partial H^{(r)}}{\partial \Phi_j}.$$

- **The solutions** are

  $$\Phi_j(t) = \Phi_j(0), \quad \varphi_j(t) = \dot{\varphi}_j(0) t + \varphi_j(0).$$
Analytical Integration

\[(\eta(0), \xi(0)) \xrightarrow{\text{Secular + NF}} (\Phi(0), \varphi(0))\]

\[(\eta(t), \xi(t)) \xleftarrow{\text{Numerical integration}} (\Phi(t), \varphi(t))\]

\[\Phi(t) = \Phi(0)\]
\[\varphi(t) = \dot{\phi}(0)t + \varphi(0)\]
**HD 134987**: the system is secular.

![Graph showing Eccentricities analysis for HD 134987](attachment://ecc_0.dat)

![Graph showing Orbitals analysis](attachment://orbitals_1.dat)
But... Near Mean Motion Resonance (HD 108874)

**HD 108874**: the system is “close” to the 4 : 1 MMR.

First order averaged Hamiltonian failed.
A mean motion resonance (MMR) appears when the revolution periods of the two planets are commensurable

\[ \frac{T_1}{T_2} = \frac{p}{q}, \quad \text{with } p, q \in \mathbb{N}. \]

For example, if we consider the 2:1 resonance,
Second Order Averaging

In order to get rid of the fast motion, instead of simply erasing the terms depending on fast angles $\lambda$, we perform a **canonical transformation via Lie Series** to kill the terms

$$\left[ \mu h^{(T)}_{0,0} \right]_{\lambda:K} (\lambda, \xi, \eta), \ldots, \left[ \mu h^{(T)}_{0,ECC} \right]_{\lambda:K} (\lambda, \xi, \eta),$$

where $[\cdot]_{\lambda:K}$ means the truncation of the harmonics of degree greater than $K$.

The parameter $ECC$ is chosen to **include in the secular model** the main effects due to the possible *proximity to a mean motion resonance*.

This is the so called **second order averaging**.
The details of the transformation

This procedure is essentially a “Kolmogorov’s like” step of normalization, we have to solve the homological equation

$$n^* \frac{\partial \chi}{\partial \lambda} + \left[ \mu \ h_0^{(T)} \right]_{\lambda:K} = 0,$$

and find the generating function $\chi$, that is of order $O(\mu)$.

$$H^{(O2)} = \exp \mathcal{L}_\chi H^{(T)} = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}_\chi^j H^{(T)},$$

this transformation, by construction, kill the term $\left[ \mu \ h_{j_1,j_2}^{(T_F)} \right]_{\lambda:K}$.

The transformed Hamiltonian still has a term of the same type, but at least of order $O(\mu^2)$.

This is the so called Hamiltonian at order two in the masses.
Analytical Integration

\[ (\eta(0), \xi(0)) \xrightarrow{\text{Secular + NF}} (\Phi(0), \varphi(0)) \]

\[ (\eta(t), \xi(t)) \xrightarrow{\text{Numerical integration}} (\Phi(t), \varphi(t)) \]

\[ \Phi(t) = \Phi(0) \]
\[ \varphi(t) = \dot{\varphi}(0) t + \varphi(0) \]
**HD 11506**: the system is “close” to the 7 : 1 MMR (weak MMR).
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**HD 108874**: the system is “close” to the 4 : 1 MMR (stronger MMR).
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**Graph:**

- Title: Eccentricities analy_HD_108874_ord2
- Data from:
  - `./ecc_0.dat`
  - `./ecc_1.dat`
  - `./orbits_1.dat` u 1:4
  - `./orbits_2.dat` u 1:4

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**Axes:**

- X-axis: T (time in units)
- Y-axis: Eccentricity (ecc)
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**Answers:**

1) If the system is not too close to a mean motion resonance, providing an approximation of the motions of the secular variables *up to order two in the masses*, the secular evolution is well approximated via Birkhoff normal form.

2) The secular Hamiltonian *at order two in the masses* is **explicitly constructed** via Lie Series, so the generating function contains the information about the **proximity to a mean motion**.
Thanks for your attention!

Questions?

Comments?