

# FOURIER–MUKAI FUNCTORS IN THE SUPPORTED CASE

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ABSTRACT. We prove that exact functors between the categories of perfect complexes supported on projective schemes are of Fourier–Mukai type if the functor satisfies a condition weaker than being fully faithful. We also get generalizations of the results in the literature in the non-supported case. Some applications are discussed and, along the way, we prove that the category of perfect supported complexes has a strongly unique enhancement.

## 1. INTRODUCTION

One of the most intriguing open questions in the theory of derived categories is whether all exact functors between the categories of perfect complexes (or between the bounded derived categories of coherent sheaves) on projective schemes are of Fourier–Mukai type. It might be worth recalling that, if  $X_1$  and  $X_2$  are projective schemes, an exact functor  $F: \mathbf{Perf}(X_1) \rightarrow \mathbf{Perf}(X_2)$  between the corresponding categories of perfect complexes is a *Fourier–Mukai functor* (or of *Fourier–Mukai type*) if there exists  $\mathcal{E} \in D^b(X_1 \times X_2)$  and an isomorphism of exact functors  $F \cong \Phi_{\mathcal{E}}$ . Here  $\Phi_{\mathcal{E}}: \mathbf{Perf}(X_1) \rightarrow \mathbf{Perf}(X_2)$  is the exact functor defined by

$$\Phi_{\mathcal{E}} := \mathbf{R}(p_2)_*(\mathcal{E} \otimes^{\mathbf{L}} p_1^*(-)),$$

where  $p_i: X_1 \times X_2 \rightarrow X_i$  is the natural projection. The complex  $\mathcal{E}$  is called a *kernel* of  $F$ .

While, in general, the kernel is certainly not unique (up to isomorphism) due to [10], the question about the existence of such kernels is widely open. Indeed, despite the fact that a conjecture in [4] would suggest a positive answer to it, for the time being, only partial results in this direction are available. Let us recall some of them. In [25] (together with [5]) the case of exact fully faithful functors between the bounded derived categories of coherent sheaves on smooth projective varieties is completely solved by Orlov. Various generalizations to quotient stacks and twisted categories were given in [19] by Kawamata and in [11] respectively. In particular, the main result of [11] shows that all exact functors  $F: D^b(X_1) \rightarrow D^b(X_2)$  such that

$$(1.1) \quad \mathrm{Hom}_{D^b(X_2)}(F(\mathcal{A}), F(\mathcal{B})[k]) = 0,$$

for any  $\mathcal{A}, \mathcal{B} \in \mathbf{Coh}(X_1)$  and any integer  $k < 0$ , are Fourier–Mukai functors and their kernels are unique, up to isomorphism.

The inspiration for our results in this paper comes from the new approach to the representability problem in [24], where the authors show that all exact fully faithful functors  $F: \mathbf{Perf}(X_1) \rightarrow \mathbf{Perf}(X_2)$  between the categories of perfect complexes on the projective schemes  $X_1$  and  $X_2$  are of Fourier–Mukai type. To show this, Lunts and Orlov prove that such fully faithful functors admit

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dg-lifts. At that point, they can invoke the representability result in [27]. Indeed, Toën proved that, in the dg-setting, all dg-(quasi)functors are of Fourier–Mukai type. Notice that the strategy in [24] allows the authors to improve the results in [2].

To make clear the categorical setting we are going to work with, let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  such that the structure sheaf  $\mathcal{O}_{iZ_1}$  of the  $i$ -th infinitesimal neighbourhood of  $Z_1$  in  $X_1$  is in  $\mathbf{Perf}(X_1)$ , for every  $i > 0$ . This last condition is verified for instance when either  $Z_1 = X_1$  or  $X_1$  is smooth. Moreover let  $X_2$  be a separated scheme of finite type over the base field  $\mathbb{k}$  with a subscheme  $Z_2$  which is proper over  $\mathbb{k}$ . One can then consider the categories  $\mathbf{Perf}_{Z_i}(X_i)$  of perfect complexes on  $X_i$  with cohomology sheaves supported on  $Z_i$ . The definition of Fourier–Mukai functor makes perfect sense also in this context (see Definition 2.2).

A rewriting of (1.1) in the supported setting which weakens the fully-faithfulness condition in [24, 25] requires a bit of care. Indeed, assuming  $X_1, X_2, Z_1$  and  $Z_2$  to be as above, one can consider exact functors  $F: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  such that

- (\*) (1)  $\mathrm{Hom}(F(\mathcal{A}), F(\mathcal{B})[k]) = 0$ , for any  $\mathcal{A}, \mathcal{B} \in \mathbf{Coh}_{Z_1}(X_1) \cap \mathbf{Perf}_{Z_1}(X_1)$  and any integer  $k < 0$ ;
- (2) For all  $\mathcal{A} \in \mathbf{Perf}_{Z_1}(X_1)$  with trivial cohomologies in (strictly) positive degrees, there is  $N \in \mathbb{Z}$  such that

$$\mathrm{Hom}(F(\mathcal{A}), F(\mathcal{O}_{|iZ_1}(jH_1))) = 0,$$

for any  $i < N$  and any  $j \ll i$ , where  $H_1$  is an ample divisor on  $X_1$ .

At first sight this condition may look a bit involved, but if  $Z_1 = X_1$  is smooth, then part (2) of (\*) is redundant and thus (\*) turns out to be equivalent to (1.1) (see Proposition 3.13). In general full functors always satisfy (\*), if we assume further that the maximal 0-dimensional torsion subsheaf  $T_0(\mathcal{O}_{Z_1})$  of  $\mathcal{O}_{Z_1}$  is trivial. Actually, due to [8], a full functor is automatically faithful if  $Z_1$  is connected. We will discuss in Section 3.4 the existence of non-full functors with property (\*).

We are now ready to state our first main result.

**Theorem 1.1.** *Let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  such that  $\mathcal{O}_{iZ_1} \in \mathbf{Perf}(X_1)$ , for all  $i > 0$ , and let  $X_2$  be a separated scheme of finite type over the base field  $\mathbb{k}$  with a subscheme  $Z_2$  which is proper over  $\mathbb{k}$ . Let*

$$F: \mathbf{Perf}_{Z_1}(X_1) \longrightarrow \mathbf{Perf}_{Z_2}(X_2)$$

be an exact functor.

If  $F$  satisfies (\*), then there exist  $\mathcal{E} \in \mathbf{D}_{Z_1 \times Z_2}^b(\mathbf{Qcoh}(X_1 \times X_2))$  and an isomorphism of functors  $F \cong \Phi_{\mathcal{E}}^s$ . Moreover, if  $X_i$  is smooth quasi-projective, for  $i = 1, 2$ , and  $\mathbb{k}$  is perfect, then  $\mathcal{E}$  is unique up to isomorphism.

The proof, contained in Sections 4 and 5.2, uses the approach via dg-categories proposed in [24]. Clearly, assuming  $X_k = Z_k$  for  $k = 1, 2$ , our result extends the one in [24] about singular projective schemes (see Corollaries 4.8 and 4.11). Notice that the symbol  $\Phi_{\mathcal{E}}^s$  stands for the ‘supported’ Fourier–Mukai functor with kernel  $\mathcal{E}$  defined precisely in (2.2).

As a consequence of the techniques developed in Sections 2.2 and 3, we can state our second main result whose proof is contained in Section 4.2.

**Theorem 1.2.** *Let  $X$  be a smooth quasi-projective scheme containing a projective subscheme  $Z$  such that  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$ , for all  $i > 0$ , and  $T_0(\mathcal{O}_Z) = 0$ . Then  $\mathbf{Perf}_Z(X)$  has a strongly unique enhancement.*

The notion of enhancement and its strong uniqueness is discussed in Section 4.1. For the moment we can roughly think of an enhancement of  $\mathbf{Perf}_Z(X)$  as a (pre-triangulated) dg-category whose homotopy category is equivalent to  $\mathbf{Perf}_Z(X)$ . The enhancement is strongly unique if two such are (quasi-)equivalent at the dg-category level and such an equivalence satisfies some additional condition. It is worth noticing that the particular case  $X = Z$  is one of the main results in [24] (see Corollary 4.6).

**Motivations.** Due to the technical nature of Theorems 1.1 and 1.2, some geometric motivations are certainly in order here. From our point of view the reason for studying exact functors between supported categories is twofold. On one side the conjecture in [4] concerning the fact that all ‘geometric’ functors are of Fourier–Mukai type appears extremely difficult to be proved in complete generality. Thus it makes sense to test its validity weakening the assumptions on the geometric nature of the triangulated categories involved and on the exact functors between them. In this sense, this paper is in the same spirit as [11] and [24].

On the other hand, one would like to study easy-to-handle *d-Calabi–Yau categories*, i.e. triangulated categories whose Serre functor is isomorphic to the shift by the positive integer  $d$ . The most challenging examples are certainly provided by the derived categories of smooth projective Calabi–Yau threefolds. Indeed, the homological version of the Mirror Symmetry conjecture for those threefolds involves these categories and the manifold parametrizing stability conditions [7] into which (up to the quotient by the group of autoequivalences) the Kähler moduli space embeds. One big open problem in this direction is the lack of examples of stability conditions for Calabi–Yau threefolds.

The group of autoequivalences of the derived category, besides being an interesting algebraic object in itself, acts on the stability manifold. Already for Calabi–Yau manifolds of dimension 2 (i.e. K3 surfaces), this group is very complicated and one of the main motivations of [25] is to give an input to its study. As for stability conditions, in higher dimension the situation becomes much more involved.

Therefore, following suggestions from the physics literature, one may start from the non-compact or the so called ‘open’ Calabi–Yau’s. Let us be more precise discussing some explicit examples where the ambient space  $X_1$  is smooth and Theorem 1.1 (or a variant of it) applies.

Following [14] and [22], one can consider the triangulated category  $\mathbf{T}_S$  generated by a  $d$ -spherical object  $S$  (here  $d$  is a positive integer). An object  $S$  is  $d$ -spherical if the graded algebra  $\mathrm{Ext}^*(S, S)$  is isomorphic to the cohomology of a  $d$ -sphere. We will study this example in Section 4.4 when  $d = 1$  as in this case  $\mathbf{T}_S$  is nothing but  $D_p^b(C)$ , where  $C$  is a smooth curve and  $p \in C$  is a closed point. Thus we obtain the following result, which is a particular case of Proposition 4.14.

**Proposition 1.3.** *Every exact autoequivalence of  $\mathbf{T}_S$  is of Fourier–Mukai type if  $S$  is a 1-spherical object.*

This completes the picture in [14] which provides a description of the subgroup of Fourier–Mukai autoequivalences. We should remark here that the result above is not a direct consequence of Theorem 1.1 as the maximal 0-dimensional torsion subsheaf of  $\mathcal{O}_p$  is obviously not trivial and part (2) of (\*) does not necessarily hold true.

Interesting examples of 2-Calabi–Yau categories are provided by the local resolutions of  $A_n$ -singularities on surfaces which were studied in [16, 17]. More precisely, one considers  $Y = \text{Spec}(\mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^{n+1}))$  (the  $A_n$ -singularity), the minimal resolution  $f: X \rightarrow Y$  and  $Z := f^{-1}(p)$ , where  $p$  is the closed point in  $Y$ . Notice that, in this case,  $T_0(\mathcal{O}_Z) = 0$ . The category one wants to consider is then  $D_Z^b(X) = \mathbf{Perf}_Z(X)$  and using Theorem 1.1 we can reprove in a direct way the following result already contained in [17].

**Corollary 1.4.** *Every exact autoequivalence of  $D_Z^b(X)$  is of Fourier–Mukai type.*

Finally, to get examples of 3-Calabi–Yau categories one can take the total space  $\text{tot}(\omega_{\mathbb{P}^2})$  of the canonical bundle of  $\mathbb{P}^2$  ([1]). In this case, if  $Z$  denotes the zero section of the projection  $\text{tot}(\omega_{\mathbb{P}^2}) \rightarrow \mathbb{P}^2$ , the derived category  $\mathbf{Perf}_Z(\text{tot}(\omega_{\mathbb{P}^2})) = D_Z^b(\text{tot}(\omega_{\mathbb{P}^2}))$  is a 3-Calabi–Yau category and may be seen as an interesting example to test predictions about Mirror Symmetry and the topology of the space of stability conditions according to Bridgeland’s definition (see [1] for results in this direction). Here again  $T_0(\mathcal{O}_Z) = 0$  and so Theorem 1.1 yields the following.

**Corollary 1.5.** *Every exact autoequivalence of  $D_Z^b(\text{tot}(\omega_{\mathbb{P}^2}))$  is of Fourier–Mukai type.*

As an application of Proposition 4.14 and Theorem 1.2, the triangulated categories in the three examples above have strongly unique enhancements.

**The plan of the paper.** In Section 2 we provide the necessary preliminary material concerning derived categories of supported sheaves, and we introduce the notion of weakly ample set. Then we prove a criterion (generalizing others present in the literature) for extending a morphism defined on the weakly ample set between exact functors satisfying (\*). This is done in Section 3 using the notion of convolution. In Section 4 we deal with the existence of Fourier–Mukai kernels and the strong uniqueness of enhancements. In particular, we need to generalize and to modify the argument in [24] to make it work in our setting. In the same section we also discuss the case of 1-spherical objects. Section 5 deals with various questions about uniqueness of Fourier–Mukai kernels.

**Notation.** In the paper,  $\mathbb{k}$  is a field. All schemes are assumed to be of finite type and separated over  $\mathbb{k}$ . All additive (in particular, triangulated) categories and all additive (in particular, exact) functors will be assumed to be  $\mathbb{k}$ -linear. An additive category will be called Hom-finite if the  $\mathbb{k}$ -vector space  $\text{Hom}(A, B)$  is finite dimensional for every objects  $A$  and  $B$ . If  $\mathbf{A}$  is an abelian (or more generally an exact) category,  $D(\mathbf{A})$  denotes the derived category of  $\mathbf{A}$  and  $D^b(\mathbf{A})$  its full subcategory of complexes with bounded cohomology. Unless clearly stated, all functors are derived even if, for simplicity, we use the same symbol for a functor and its derived version. Natural

transformations (in particular, isomorphisms) between exact functors are always assumed to be compatible with shifts.

## 2. PRELIMINARIES

The first part of this section provides a quick introduction to some basic and well-known facts concerning the derived categories of supported sheaves. Then we define and discuss the notion of weakly ample set.

**2.1. Supported categories.** Let  $X$  be a separated scheme of finite type over  $\mathbb{k}$  and let  $Z$  be a subscheme of  $X$  which is proper over  $\mathbb{k}$ . We denote by  $D_Z(\mathbf{Qcoh}(X))$  the derived category of unbounded complexes of quasi-coherent sheaves on  $X$  with cohomologies supported on  $Z$ . We will be particularly interested in the triangulated categories

$$(2.1) \quad \begin{aligned} D_Z^b(\mathbf{Qcoh}(X)) &:= D_Z(\mathbf{Qcoh}(X)) \cap D^b(\mathbf{Qcoh}(X)) \\ D_Z^b(X) &:= D_Z(\mathbf{Qcoh}(X)) \cap D^b(X), \end{aligned}$$

where  $D^b(X) := D_{\mathbf{Coh}}^b(\mathbf{Qcoh}(X))$  is the full subcategory of  $D^b(\mathbf{Qcoh}(X))$  consisting of complexes with coherent cohomologies. Denote by  $\mathbf{Perf}(X)$  the category of perfect complexes on  $X$ , coinciding with the category of compact objects in  $D(\mathbf{Qcoh}(X))$ . Notice that  $\mathbf{Perf}(X) \subseteq D^b(X)$  and, if  $X$  is quasi-projective, equality holds if and only if  $X$  is regular. In the supported case we set

$$\mathbf{Perf}_Z(X) := D_Z(\mathbf{Qcoh}(X)) \cap \mathbf{Perf}(X).$$

Thus, if  $X$  is smooth,  $\mathbf{Perf}_Z(X) = D_Z^b(X)$ . We denote by  $T_0(\mathcal{O}_Z)$  the maximal 0-dimensional torsion subsheaf of  $\mathcal{O}_Z$ .

**Proposition 2.1.** ([26], **Theorem 6.8.**) *The category  $D_Z(\mathbf{Qcoh}(X))$  is compactly generated and the category of compact objects  $D_Z(\mathbf{Qcoh}(X))^c$  coincides with  $\mathbf{Perf}_Z(X)$ .*

Recall that an object  $A$  in a triangulated category  $\mathbf{T}$  is *compact* if, for each family of objects  $\{X_i\}_{i \in I} \subset \mathbf{T}$  such that  $\bigoplus_i X_i$  exists in  $\mathbf{T}$ , the canonical map

$$\bigoplus_i \mathrm{Hom}(A, X_i) \longrightarrow \mathrm{Hom}(A, \bigoplus_i X_i)$$

is an isomorphism. Moreover,  $\mathbf{T}$  is *compactly generated* if there is a set  $S$  of objects in the subcategory  $\mathbf{T}^c$  of compact objects of  $\mathbf{T}$  such that, given  $E \in \mathbf{T}$  with  $\mathrm{Hom}(A, E[i]) = 0$  for all  $A \in S$  and all  $i \in \mathbb{Z}$ , then  $E = 0$ . For more details, the reader can consult [26, Sect. 3.1].

The category  $D_Z(\mathbf{Qcoh}(X))$  is a full subcategory of  $D(\mathbf{Qcoh}(X))$  and let

$$\iota: D_Z(\mathbf{Qcoh}(X)) \longrightarrow D(\mathbf{Qcoh}(X))$$

be the inclusion. We use the same symbol to denote the inclusion functor for the other categories in (2.1). As long as this is not confusing and to make the notation simpler, we write  $\iota$  for the inclusions corresponding to different pairs of schemes  $(X_1, Z_1)$  and  $(X_2, Z_2)$  as above. Moreover the inclusion for the product of two schemes is denoted by  $\iota \times \iota$ .

According to [23, Sect. 3], the functor  $\iota$  has a right adjoint

$$\iota^!: D(\mathbf{Qcoh}(X)) \rightarrow D_Z(\mathbf{Qcoh}(X)) \quad \iota^!(\mathcal{E}) := \operatorname{colim}_{\rightarrow n} \mathbf{R}\mathcal{H}om(\mathcal{O}_{nZ}, \mathcal{E}),$$

where  $nZ$  is the  $n$ -th infinitesimal neighborhood of  $Z$  in  $X$  (see [23, Prop. 3.2.2]). Due to [23, Cor. 3.1.4], the functor  $\iota^!$  sends bounded complexes to bounded complexes and  $\iota^! \circ \iota \cong \text{id}$ .

Now, let  $X_1$  and  $X_2$  be separated schemes of finite type over  $\mathbb{k}$  containing, respectively, two subschemes  $Z_1$  and  $Z_2$  which are proper over  $\mathbb{k}$ .

**Definition 2.2.** An exact functor

$$F: D_{Z_1}(\mathbf{Qcoh}(X_1)) \rightarrow D_{Z_2}(\mathbf{Qcoh}(X_2))$$

is a *Fourier–Mukai functor* if there exists  $\mathcal{E} \in D_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$  and an isomorphism of exact functors

$$(2.2) \quad F \cong \Phi_{\mathcal{E}}^s := \iota^!(p_2)_*((\iota \times \iota)\mathcal{E} \otimes p_1^*(\iota(-)))$$

where  $p_i: X_1 \times X_2 \rightarrow X_i$  is the projection.

Observe that, as  $(p_2)_*((\iota \times \iota)\mathcal{E} \otimes p_1^*(\iota(-)))$  is supported on  $Z_2$ ,  $\iota \circ \Phi_{\mathcal{E}}^s \cong (p_2)_*((\iota \times \iota)\mathcal{E} \otimes p_1^*(\iota(-)))$ . Analogous definitions can be given for functors defined between bounded derived categories of quasi-coherent, coherent or perfect complexes. The object  $\mathcal{E}$  is called *Fourier–Mukai kernel*. We will use the standard notation  $\Phi_{\mathcal{E}}$  when  $Z_i = X_i$  or to denote Fourier–Mukai functors between  $D^b(X_1)$  and  $D^b(X_2)$ .

Consider the abelian categories  $\mathbf{Qcoh}_Z(X)$  and  $\mathbf{Coh}_Z(X)$  consisting of quasi-coherent and, respectively, coherent sheaves supported on  $Z$ . The following will be implicitly used at many points of this paper.

**Proposition 2.3.** ([3, Lemma 3.3, Corollary 3.4.]) *The natural functors  $D(\mathbf{Qcoh}_Z(X)) \rightarrow D_Z(\mathbf{Qcoh}(X))$  and  $D^b(\mathbf{Coh}_Z(X)) \rightarrow D_Z^b(X)$  are equivalences.*

Notice that the proof in [3] works in our generality as well. Denoting by  $i: Z \hookrightarrow X$  the closed embedding, we will also need the following result.

**Proposition 2.4.** ([3, Lemma 3.6, Corollary 3.7.]) (i) *The functor  $i_*$  is exact and the image of  $i_*$  generates  $\mathbf{Coh}_Z(X)$  as an abelian category, i.e. the smallest abelian subcategory of  $\mathbf{Coh}_Z(X)$  closed under extensions and containing the essential image of  $i_*$  is  $\mathbf{Coh}_Z(X)$  itself. Similarly, the smallest abelian subcategory closed under extensions and arbitrary direct sums containing the image of  $i_*$  in  $\mathbf{Qcoh}_Z(X)$  is  $\mathbf{Qcoh}_Z(X)$  itself.*

(ii) *The image of  $D^b(Z)$  under  $i_*$  classically generates  $D_Z^b(X)$  and the image under  $i_*$  of  $D(\mathbf{Qcoh}(Z))$  classically completely generates  $D_Z(\mathbf{Qcoh}(X))$ .*

Recall that a subcategory  $\mathbf{S}$  of a triangulated category  $\mathbf{T}$  *classically generates*  $\mathbf{T}$  if the smallest thick triangulated subcategory of  $\mathbf{T}$  containing  $\mathbf{S}$  is  $\mathbf{T}$  itself. On the other hand,  $\mathbf{S}$  *classically completely generates*  $\mathbf{T}$  if  $\mathbf{T}$  is the smallest thick subcategory which is closed under direct sums and contains  $\mathbf{S}$ .

As a matter of notation, if  $\mathbf{T}$  is a triangulated category with arbitrary direct sums and  $\mathbf{L}$  is a localizing subcategory of  $\mathbf{T}$ , we denote by  $\pi: \mathbf{T} \rightarrow \mathbf{T}/\mathbf{L}$  the natural projection functor. Recall that

a strictly full triangulated subcategory  $\mathbf{S}$  of a triangulated category  $\mathbf{T}$  is *localizing* if it is closed under arbitrary direct sums. Moreover, for an object

$$A := \{\cdots \rightarrow A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \rightarrow \cdots\}$$

in  $\mathbf{D}(\mathbf{A})$ , for  $\mathbf{A}$  an abelian category, we can consider the *gentle truncation*  $\tau_{\leq i}A$  and the *stupid truncation*  $\sigma_{\leq i}A$ . Those are, respectively, the complexes

$$\begin{aligned} \tau_{\leq i}A &:= \{\cdots \rightarrow A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} \ker d^i \rightarrow 0 \rightarrow \cdots\} \\ \sigma_{\leq i}A &:= \{\cdots \rightarrow A^j \xrightarrow{d^j} A^{j+1} \xrightarrow{d^{j+1}} \cdots \xrightarrow{d^{i-1}} A^i \rightarrow 0 \rightarrow \cdots\}. \end{aligned}$$

One can define  $\tau_{\geq i}A$  and  $\sigma_{\geq i}A$  or the related versions with  $<$  and  $>$  instead of  $\leq$  and  $\geq$  in a completely analogous way.

**2.2. Weakly ample sets.** The attempt of this section is to provide a generalization of the notion of weakly ample sequence in [17, Appendix A.2] which, in turn, is a generalization of the usual definition of ample sequence (see, for example, [25]).

**Definition 2.5.** Given a Hom-finite abelian category  $\mathbf{A}$  and a set  $I$ , a subset  $\{P_i\}_{i \in I} \subseteq \mathbf{A}$  is a *weakly ample set* if, for any  $A \in \mathbf{A}$ , there exists an integer  $i \in I$  such that:

- (1) The natural morphism  $\mathrm{Hom}_{\mathbf{A}}(P_i, A) \otimes P_i \rightarrow A$  is surjective;
- (2) There is a natural number  $k$  and an epimorphism  $P_i^{\oplus k} \rightarrow A$  such that the induced morphism

$$\mathrm{Hom}_{\mathbf{A}}(A, A[m]) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(P_i^{\oplus k}, A[m])$$

is trivial for all integers  $m \neq 0$ ;

- (3)  $\mathrm{Hom}_{\mathbf{A}}(A, P_i) = 0$ .

**Remark 2.6.** In analogy with the case of (weakly) ample sequences, a weakly ample set is simply called an *ample set* if (2) is replaced by the stronger condition  $\mathrm{Hom}_{\mathbf{A}}(P_i, A[m]) = 0$ , for all  $m \neq 0$ . Observe that every (weakly) ample sequence is a (weakly) ample set.

To provide examples of weakly ample sets which are suited for the supported setting we are working in, let  $X$  be a quasi-projective scheme and let  $Z$  be a projective subscheme of  $X$ . Assume further that  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$ , for all  $i > 0$ . Take  $H$  an ample divisor on  $X$  and define the subset of  $\mathbf{Coh}_Z(X)$

$$(2.3) \quad \mathbf{Amp}(Z, X, H) := \{\mathcal{O}_{iZ}(jH)\}_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}.$$

When needed, we will think of  $\mathbf{Amp}(Z, X, H)$  as the corresponding full subcategory of  $\mathbf{Coh}_Z(X)$ .

**Example 2.7.** There are two interesting geometric situations for which  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$ , for all  $i > 0$ , and thus  $\mathbf{Amp}(Z, X, H)$  is contained in  $\mathbf{Perf}_Z(X)$ . Namely one can take  $X$  to be a quasi-projective scheme containing a projective subscheme  $Z$  such that either  $Z = X$  or  $X$  is smooth.

The following result will be essential for the rest of the paper.

**Proposition 2.8.** *Assume that  $X$ ,  $Z$  and  $H$  are as above. Then  $\mathbf{Amp}(Z, X, H)$  satisfies (1) and (2) in Definition 2.5, and provides a set of (compact) generators of the Grothendieck category  $\mathbf{Qcoh}_Z(X)$ . Moreover, if  $T_0(\mathcal{O}_Z) = 0$ , then  $\mathbf{Amp}(Z, X, H)$  is a weakly ample set in  $\mathbf{Coh}_Z(X)$ .*

More precisely, for any  $\mathcal{A} \in \mathbf{Coh}_Z(X)$ , there is  $N \in \mathbb{Z}$  such that any  $\mathcal{O}_{|i|Z}(jH)$  with  $i < N$  and  $j \ll i$  satisfies (1), (2) (and (3) if  $T_0(\mathcal{O}_Z) = 0$ ) in Definition 2.5.

*Proof.* Notice that under the assumption  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$  for all  $i > 0$ , the objects  $P_{i,j} := \mathcal{O}_{|i|Z}(jH)$  are compact as well for all  $i, j \in \mathbb{Z}$ .

Let  $\mathcal{E}$  be a sheaf in  $\mathbf{Coh}_Z(X)$ . To prove property (1), observe that there is an integer  $n_1 < 0$  such that  $\mathcal{E}$  is an  $\mathcal{O}_{|n_1|Z}$ -module. Hence for  $i < n_1$  and  $j \ll i$  the morphism

$$\mathrm{Hom}(\mathcal{O}_{|i|Z}(jH), \mathcal{E}) \otimes \mathcal{O}_{|i|Z}(jH) \longrightarrow \mathcal{E}$$

is surjective.

To prove (2), given  $k \neq 0$ , fix a basis  $B_k := \{e_1^k, \dots, e_{m_k}^k\}$  of  $\mathrm{Hom}_{\mathbf{Coh}_Z(X)}(\mathcal{E}, \mathcal{E}[k])$ . Pick  $e_s^k \in B_k$  and consider the corresponding extension

$$\mathcal{G} \longrightarrow \mathcal{E} \xrightarrow{e_s^k} \mathcal{E}[k]$$

which we may assume to be an extension in the category of complexes of  $\mathcal{O}_{|n_s^k|Z}$ -modules, for some  $n_s^k \ll 0$ . Since  $H$  is ample, for any  $i < n_s^k$  and  $j \ll i$ , there is a surjective morphism  $\phi: P_{i,j}^{\oplus l} \rightarrow \mathcal{E}$  and

$$\mathrm{Hom}_{|i|Z}(P_{i,j}, \mathcal{E}[k]) = 0.$$

Hence, the induced extension

$$\mathcal{G}' \longrightarrow P_{i,j}^{\oplus l} \xrightarrow{e_s^k \circ \phi} \mathcal{E}[k]$$

is trivial in  $|i|Z$ , for  $i < n_s^k$ , and so in  $D_Z^b(X)$ . Now it is enough to repeat the same argument for the finite number of  $0 \neq k$ 's for which  $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[k]) \neq 0$  and any element in the basis  $B_k$ , taking  $n_2 := \min\{n_1^k, \dots, n_{m_k}^k\}_{0 \neq k \in \mathbb{Z}}$ .

Assuming  $T_0(\mathcal{O}_Z) = 0$  (which, indeed, implies  $T_0(\mathcal{O}_{|i|Z}) = 0$ ), property (3) is easily verified taking  $i < n_1$  and  $j \ll i$ . Finally set  $N := \min\{n_1, n_2\}$ .

To prove that  $\mathbf{Amp}(Z, X, H)$  is a set of generators for the category  $\mathbf{Qcoh}_Z(X)$ , it is enough to observe that, in view of (1) of Definition 2.5 and the fact that any quasi-coherent sheaf is the direct limit of its coherent subsheaves, for any  $\mathcal{E} \in \mathbf{Qcoh}_Z(X)$  there is a surjection  $\bigoplus_{j \in S} P_j \rightarrow \mathcal{E}$  where  $S$  is a set and  $P_j \in \mathbf{Amp}(Z, X, H)$  for all  $j \in S$  (see [18, Sect. 8.3]).  $\square$

**Example 2.9.** If  $X$  is the resolution of an  $A_n$ -singularity and  $Z$  is the exceptional locus, a special case of weakly ample set for  $\mathbf{Coh}_Z(X)$  is provided by the *weak ample sequence*  $\mathbf{C}$  in [17, Appendix A]. Recall that  $\mathbf{C} = \{\mathcal{O}_{|i|Z}(iH) \in \mathbf{Amp}(Z, X, H) : i \in \mathbb{Z}\}$ .

### 3. EXTENDING NATURAL TRANSFORMATIONS

In this section we deal with the second key ingredient in our proof, namely a criterion to extend natural transformations (in particular, isomorphisms) between functors. We tried to put this result in a generality which goes beyond the scope of this paper but which may be useful in future works (see, for example, [9, Prop. 5.15]).

**3.1. Convolutions.** In this section we collect some well-known facts about convolutions which will be used in the paper. Most of the terminology is taken from [19, 25] (see also [11]).

A bounded complex in a triangulated category  $\mathbf{T}$  is a sequence of objects and morphisms in  $\mathbf{T}$

$$(3.1) \quad A_m \xrightarrow{d_m} A_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} A_0$$

such that  $d_j \circ d_{j+1} = 0$  for  $0 < j < m$ . A *right convolution* of (3.1) is an object  $A$  together with a morphism  $d_0: A_0 \rightarrow A$  such that there exists a diagram in  $\mathbf{T}$

$$\begin{array}{ccccccc} A_m & \xrightarrow{d_m} & A_{m-1} & \xrightarrow{d_{m-1}} & \cdots & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & A_0 \\ & \searrow \text{id} & \circlearrowleft & \searrow & \circlearrowleft & \searrow & \circlearrowleft & \searrow & \searrow d_0 \\ & A_m & & C_{m-1} & & \cdots & & C_1 & & A \\ & & \xleftarrow{[1]} & \xleftarrow{[1]} & \cdots & \xleftarrow{[1]} & \xleftarrow{[1]} & \xleftarrow{[1]} & & \end{array}$$

where the triangles with a  $\circlearrowleft$  are commutative and the others are distinguished.

Let  $d_0: A_0 \rightarrow A$  be a right convolution of (3.1). If  $\mathbf{T}'$  is another triangulated category and  $G: \mathbf{T} \rightarrow \mathbf{T}'$  is an exact functor, then  $G(d_0): G(A_0) \rightarrow G(A)$  is a right convolution of

$$G(A_m) \xrightarrow{G(d_m)} G(A_{m-1}) \xrightarrow{G(d_{m-1})} \cdots \xrightarrow{G(d_1)} G(A_0).$$

The following results will be used in the rest of this section.

**Lemma 3.1.** ([19], Lemmas 2.1 and 2.4.) *Let (3.1) be a complex in  $\mathbf{T}$  satisfying*

$$(3.2) \quad \text{Hom}_{\mathbf{T}}(A_a, A_b[r]) = 0 \text{ for any } a > b \text{ and } r < 0.$$

*Then (3.1) has a right convolution which is uniquely determined up to isomorphism (in general non canonical).*

**Lemma 3.2.** ([11], Lemma 3.3.) *Let*

$$\begin{array}{ccccccc} A_m & \xrightarrow{d_m} & A_{m-1} & \xrightarrow{d_{m-1}} & \cdots & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & A_0 \\ \downarrow f_m & & \downarrow f_{m-1} & & & & \downarrow f_1 & & \downarrow f_0 \\ B_m & \xrightarrow{e_m} & B_{m-1} & \xrightarrow{e_{m-1}} & \cdots & \xrightarrow{e_2} & B_1 & \xrightarrow{e_1} & B_0 \end{array}$$

*be a morphism of complexes both satisfying (3.2) and such that*

$$\text{Hom}_{\mathbf{T}}(A_a, B_b[r]) = 0 \text{ for any } a > b \text{ and } r < 0.$$

*Assume that the corresponding right convolutions are of the form  $(d_0, 0): A_0 \rightarrow A \oplus \bar{A}$  and  $(e_0, 0): B_0 \rightarrow B \oplus \bar{B}$  and that  $\text{Hom}_{\mathbf{T}}(A_p, B[r]) = 0$  for  $r < 0$  and any  $p$ . Then there exists a unique morphism  $f: A \rightarrow B$  such that  $f \circ d_0 = e_0 \circ f_0$ . If moreover each  $f_i$  is an isomorphism, then  $f$  is an isomorphism as well.*

Let  $\mathbf{T} := D^b(\mathbf{A})$  for some abelian category  $\mathbf{A}$  and let  $E$  be a complex as in (3.1) and such that every  $A_i$  is an object of  $\mathbf{A}$ . Then a right convolution of  $E$  (which is unique up to isomorphism by Lemma 3.1) is the natural morphism  $A_0 \rightarrow E^\bullet$ , where  $E^\bullet$  is the object of  $D^b(\mathbf{A})$  naturally associated to  $E$  (namely,  $E^i := A_{-i}$  for  $-m \leq i \leq 0$  and otherwise  $E^i := 0$ , with differential  $d_{-i}: E^i \rightarrow E^{i+1}$  for  $-m \leq i < 0$ ).

**3.2. The criterion: extension to a subcategory.** Looking carefully at the proof of [11, Prop. 3.7], one sees that the notion of ample sequence can be replaced there by the one of weakly ample set. In particular, if  $\mathbf{T}$  is a triangulated category and  $\mathbf{A}$  is a Hom-finite abelian category, we can deal with functors  $F: D^b(\mathbf{A}) \rightarrow \mathbf{T}$  satisfying the following condition:

$$(\diamond) \quad \text{Hom}_{\mathbf{T}}(F(A), F(B)[k]) = 0, \text{ for any } A, B \in \mathbf{A} \text{ and any } k < 0.$$

Hence one can prove the following result.

**Proposition 3.3.** *Let  $\mathbf{T}$  be a triangulated category and let  $\mathbf{A}$  be a Hom-finite abelian category of finite homological dimension. Assume that  $\{P_i\}_{i \in I} \subseteq \mathbf{A}$  is a weakly ample set and denote by  $\mathbf{C}$  the corresponding full subcategory. Let  $F_1, F_2: D^b(\mathbf{A}) \rightarrow \mathbf{T}$  be exact functors and let  $f: F_1|_{\mathbf{C}} \xrightarrow{\sim} F_2|_{\mathbf{C}}$  be an isomorphism of functors. Assume moreover the following:*

- (i) *the functor  $F_1$  satisfies  $(\diamond)$ ;*
- (ii)  *$F_1$  has a left adjoint.*

*Then there exists an isomorphism of exact functors  $g: F_1 \xrightarrow{\sim} F_2$  extending  $f$ .*

In the rest of this paper we would like to apply Proposition 3.3 but, unfortunately, in the supported case a functor  $F: D_{Z_1}^b(X_1) \rightarrow D_{Z_2}^b(X_2)$  may not have left or right adjoint. Thus we are going to prove a more general result (Proposition 3.7, whose proof is however much inspired by those of [25, Prop. 2.16] and [11, Prop. 3.7]), from which Proposition 3.3 will follow easily (see the end of Section 3.3). To this purpose, we first introduce the categorical setting which will be used in the rest of Section 3.

Indeed, to weaken condition  $(\diamond)$ , let  $\mathbf{E}$  be a full exact subcategory of a Hom-finite abelian category  $\mathbf{A}$  satisfying the following conditions:

- (E1) A morphism in  $\mathbf{E}$  is an admissible epimorphism if and only if it is an epimorphism in  $\mathbf{A}$ ;
- (E2) There is a set  $\{P_i\}_{i \in I} \subseteq \mathbf{E}$  which satisfies properties (1) and (2) of Definition 2.5;
- (E3) For all  $A \in D^b(\mathbf{E}) \cap \mathbf{A}$ , there exists an integer  $N(A)$  such that  $\text{Hom}_{D^b(\mathbf{A})}(A, B[i]) = 0$ , for every  $i > N(A)$  and every  $B \in D^b(\mathbf{E}) \cap \mathbf{A}$ .

The reader who is not familiar with the language of exact categories can have a look at [20] (where admissible epimorphisms are called deflations).

**Remark 3.4.** Under conditions (E1) and (E2),  $D^b(\mathbf{E})$  can be identified with a full subcategory of  $D^b(\mathbf{A})$  by [20, Thm. 12.1] (or rather its dual version). Notice that, by (1) of Definition 2.5, for each object  $E$  of  $\mathbf{E}$  there is an epimorphism  $A \twoheadrightarrow E$  in  $\mathbf{A}$ .

For  $\mathbf{E}$  and  $\mathbf{A}$  satisfying (E1) and (E2), we will consider exact functors  $F: D^b(\mathbf{E}) \rightarrow \mathbf{T}$  (for  $\mathbf{T}$  a triangulated category) such that

- ( $\Delta$ ) (1)  $\text{Hom}(F(A), F(B)[k]) = 0$ , for any  $A, B \in D^b(\mathbf{E}) \cap \mathbf{A}$  and any integer  $k < 0$ ;
- (2) For all  $C \in D^b(\mathbf{E})$  with cohomologies in non-positive degrees, there is  $i \in I$  such that

$$\text{Hom}(F(C), F(P_i)) = 0$$

and  $i$  satisfies properties (1) and (2) of Definition 2.5 for  $H^0(C)$ .

In order to state our first extension result, we need some more notation. Let  $\mathbf{C}$  be the full subcategory of  $\mathbf{A}$  with objects  $\{P_i\}_{i \in I}$  and set  $\mathbf{D}_0$  to be the (strictly) full subcategory of  $D^b(\mathbf{E})$  whose objects are isomorphic to shifts of objects of  $\mathbf{A}$ .

**Proposition 3.5.** *Let  $\mathbf{T}$  be a triangulated category and let  $\mathbf{E}$  be a full exact subcategory of a Hom-finite abelian category  $\mathbf{A}$  satisfying (E1), (E2) and (E3). Let  $F_1, F_2: D^b(\mathbf{E}) \rightarrow \mathbf{T}$  be exact functors with a natural transformation  $f: F_1|_{\mathbf{C}} \rightarrow F_2|_{\mathbf{C}}$ . Assume moreover the following:*

- (i)  $F_1$  and  $F_2$  both satisfy condition (1) of  $(\Delta)$  and

$$\mathrm{Hom}(F_1(A), F_2(B)[k]) = 0,$$

for any  $A, B \in D^b(\mathbf{E}) \cap \mathbf{A}$  and any integer  $k < 0$ .

Then there exists a unique natural transformation compatible with shifts  $f_0: F_1|_{\mathbf{D}_0} \rightarrow F_2|_{\mathbf{D}_0}$  extending  $f$ .

*Proof.* For any  $i \in I$ , let  $f_i := f(P_i): F_1(P_i) \rightarrow F_2(P_i)$ . We also set  $\mathbf{F} := D^b(\mathbf{E}) \cap \mathbf{A}$ .

The first step consists in showing that  $f$  extends uniquely to a natural transformation  $F_1|_{\mathbf{F}} \rightarrow F_2|_{\mathbf{F}}$ . To this purpose, one starts with  $A \in \mathbf{F}$  and takes a(n infinite) resolution

$$(3.3) \quad \dots \rightarrow P_{i_j}^{\oplus k_j} \xrightarrow{d_j} P_{i_{j-1}}^{\oplus k_{j-1}} \xrightarrow{d_{j-1}} \dots \xrightarrow{d_1} P_{i_0}^{\oplus k_0} \xrightarrow{d_0} A \rightarrow 0,$$

where  $i_j \in I$  and  $k_j \in \mathbb{N}$  for every  $j \in \mathbb{N}$ . Notice that this is possible thanks to condition (1) of Definition 2.5. Let  $N(A)$  be as in (E3), fix  $m > N(A)$  and consider the bounded complex

$$R_m := \{P_{i_m}^{\oplus k_m} \xrightarrow{d_m} P_{i_{m-1}}^{\oplus k_{m-1}} \xrightarrow{d_{m-1}} \dots \xrightarrow{d_1} P_{i_0}^{\oplus k_0}\}.$$

It is easy to see that a (unique up to isomorphism) convolution of  $R_m$  is  $(d_0, 0): P_{i_0}^{\oplus k_0} \rightarrow A \oplus K_m[m]$ , where  $K_m := \ker d_m \in \mathbf{A}$ . Indeed, we can think of  $R_m$  as an object in  $D^b(\mathbf{E})$  sitting in a distinguished triangle

$$K_m[m] \longrightarrow R_m \longrightarrow A \longrightarrow K_m[m+1]$$

in  $D^b(\mathbf{E})$ , and so  $K_m \in \mathbf{F}$ . To conclude, observe that, due to the choice of  $m$ , we have

$$\mathrm{Hom}_{D^b(\mathbf{A})}(A, K_m[m+1]) \cong \mathrm{Hom}_{D^b(\mathbf{A})}(A, K_m[m]) \cong 0.$$

Hence for  $q \in \{1, 2\}$  the complex

$$F_q(R_m) := \{F_q(P_{i_m}^{\oplus k_m}) \xrightarrow{F_q(d_m)} F_q(P_{i_{m-1}}^{\oplus k_{m-1}}) \xrightarrow{F_q(d_{m-1})} \dots \xrightarrow{F_q(d_1)} F_q(P_{i_0}^{\oplus k_0})\}$$

admits a convolution  $(F_q(d_0), 0): F_q(P_{i_0}^{\oplus k_0}) \rightarrow F_q(A \oplus K_m[m])$ . Lemma 3.1 and condition (i) ensure that such a convolution is unique up to isomorphism. Moreover, again by (i),

$$\mathrm{Hom}_{\mathbf{T}}(F_1(P_{i_j}), F_2(P_{i_k})[r]) \cong \mathrm{Hom}_{\mathbf{T}}(F_1(P_{i_i}), F_2(A)[r]) \cong 0$$

for any  $i_j, i_k, i_i \in \{i_0, \dots, i_m\}$  and  $r < 0$ . Hence we can apply Lemma 3.2 getting a unique morphism  $f_A: F_1(A) \rightarrow F_2(A)$  making the following diagram commutative:

$$\begin{array}{ccccccccc} F_1(P_{i_m}^{\oplus k_m}) & \xrightarrow{F_1(d_m)} & F_1(P_{i_{m-1}}^{\oplus k_{m-1}}) & \xrightarrow{F_1(d_{m-1})} & \dots & \xrightarrow{F_1(d_1)} & F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & F_1(A) \\ \downarrow f_{i_m}^{\oplus k_m} & & \downarrow f_{i_{m-1}}^{\oplus k_{m-1}} & & & & \downarrow f_{i_0}^{\oplus k_0} & & \downarrow f_A \\ F_2(P_{i_m}^{\oplus k_m}) & \xrightarrow{F_2(d_m)} & F_2(P_{i_{m-1}}^{\oplus k_{m-1}}) & \xrightarrow{F_2(d_{m-1})} & \dots & \xrightarrow{F_2(d_1)} & F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A). \end{array}$$

By Lemma 3.2, the definition of  $f_A$  does not depend on the choice of  $m$ . In other words, if we choose a different  $m' > N(A)$  and we truncate (3.3) in position  $m'$ , the bounded complexes  $F_q(R_{m'})$  give rise to the same morphism  $f_A$ .

To show that the definition of  $f_A$  does not depend on the choice of the resolution (3.3), consider another resolution of  $A$

$$(3.4) \quad \cdots \rightarrow P_{i'_j}^{\oplus k'_j} \xrightarrow{d'_j} P_{i'_{j-1}}^{\oplus k'_{j-1}} \xrightarrow{d'_{j-1}} \cdots \xrightarrow{d'_1} P_{i'_0}^{\oplus k'_0} \xrightarrow{d'_0} A \rightarrow 0.$$

and denote by  $f'_A: F_1(A) \rightarrow F_2(A)$  the induced morphism. In order to see that  $f_A = f'_A$ , we start by proving that there exists a third resolution

$$(3.5) \quad \cdots \rightarrow P_{i''_j}^{\oplus k''_j} \xrightarrow{d''_j} P_{i''_{j-1}}^{\oplus k''_{j-1}} \xrightarrow{d''_{j-1}} \cdots \xrightarrow{d''_1} P_{i''_0}^{\oplus k''_0} \xrightarrow{d''_0} A \rightarrow 0$$

and morphisms  $s_j: P_{i''_j}^{\oplus k''_j} \rightarrow P_{i'_j}^{\oplus k'_j}$  and  $t_j: P_{i''_j}^{\oplus k''_j} \rightarrow P_{i''_{j-1}}^{\oplus k''_{j-1}}$ , for any  $j \geq 0$ , fitting into the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d'_{j+1}} & P_{i'_j}^{\oplus k'_j} & \xrightarrow{d'_j} & P_{i'_{j-1}}^{\oplus k'_{j-1}} & \xrightarrow{d'_{j-1}} & \cdots \xrightarrow{d'_1} P_{i'_0}^{\oplus k'_0} & \xrightarrow{d'_0} & A \\ & & \uparrow t_j & & \uparrow t_{j-1} & & \uparrow t_0 & \searrow d'_0 & \\ \cdots & \xrightarrow{d''_{j+1}} & P_{i''_j}^{\oplus k''_j} & \xrightarrow{d''_j} & P_{i''_{j-1}}^{\oplus k''_{j-1}} & \xrightarrow{d''_{j-1}} & \cdots \xrightarrow{d''_1} P_{i''_0}^{\oplus k''_0} & \xrightarrow{d''_0} & A \\ & & \downarrow s_j & & \downarrow s_{j-1} & & \downarrow s_0 & \nearrow d_0 & \\ \cdots & \xrightarrow{d_{j+1}} & P_{i_j}^{\oplus k_j} & \xrightarrow{d_j} & P_{i_{j-1}}^{\oplus k_{j-1}} & \xrightarrow{d_{j-1}} & \cdots \xrightarrow{d_1} P_{i_0}^{\oplus k_0} & & \end{array}$$

In fact we are just going to show how to provide  $i''_0$ ,  $k''_0$  and the morphisms  $d''_0$ ,  $s_0$  and  $t_0$ : the rest of the construction will then follow by the same argument. Consider the short exact sequences

$$0 \rightarrow K \rightarrow P_{i_0}^{\oplus k_0} \xrightarrow{d_0} A \rightarrow 0, \quad 0 \rightarrow K' \rightarrow P_{i'_0}^{\oplus k'_0} \xrightarrow{d'_0} A \rightarrow 0$$

and, according to parts (1) and (2) of Definition 2.5 (applied to the object  $A \oplus K \oplus K'$  of  $\mathbf{A}$ ), take  $i''_0 \in I$  with an epimorphism  $d''_0: P_{i''_0}^{\oplus k''_0} \rightarrow A$ , for some  $k''_0 \in \mathbb{N}$ , and such that the induced morphism

$$\phi: \text{Hom}(A, K[1]) \rightarrow \text{Hom}(P_{i''_0}^{\oplus k''_0}, K[1]), \quad \phi': \text{Hom}(A, K'[1]) \rightarrow \text{Hom}(P_{i''_0}^{\oplus k''_0}, K'[1])$$

are trivial. From the commutative diagram with exact rows

$$\begin{array}{ccccc} \text{Hom}(A, P_{i_0}^{\oplus k_0}) & \xrightarrow{d_0 \circ (-)} & \text{Hom}(A, A) & \xrightarrow{l_1} & \text{Hom}(A, K[1]) \\ (-) \circ d''_0 \downarrow & & \downarrow (-) \circ d''_0 & & \downarrow \phi = 0 \\ \text{Hom}(P_{i''_0}^{\oplus k''_0}, P_{i_0}^{\oplus k_0}) & \xrightarrow{d_0 \circ (-)} & \text{Hom}(P_{i''_0}^{\oplus k''_0}, A) & \xrightarrow{l_2} & \text{Hom}(P_{i''_0}^{\oplus k''_0}, K[1]) \end{array}$$

we deduce that  $l_2(d''_0) = \phi(l_1(\text{id})) = 0$ , hence there exists  $s_0: P_{i''_0}^{\oplus k''_0} \rightarrow P_{i_0}^{\oplus k_0}$  such that  $d_0 \circ s_0 = d''_0$ .

In a completely similar way one finds  $t_0: P_{i''_0}^{\oplus k''_0} \rightarrow P_{i'_0}^{\oplus k'_0}$  such that  $d'_0 \circ t_0 = d''_0$ .

Denoting by  $f''_A: F_1(A) \rightarrow F_2(A)$  the morphism constructed using (3.5), we get a diagram

$$\begin{array}{ccccc}
F_1(P_{i_0}^{\oplus k''_0}) & \xrightarrow{F_1(d''_0)} & & & F_1(A) \\
\downarrow f_{i_0}^{\oplus k''_0} & \searrow F_1(s_0) & & & \downarrow f''_A \\
& & F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & F_1(A) \\
& & \downarrow f_{i_0}^{\oplus k_0} & & \downarrow f_A \quad \star \\
& & F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A) \\
& \nearrow F_2(s_0) & & & \nearrow \text{id} \\
F_2(P_{i_0}^{\oplus k''_0}) & \xrightarrow{F_2(d''_0)} & & & F_2(A)
\end{array}$$

where all squares but  $\star$  are commutative. Due to hypothesis (i) and Lemma 3.2 there exists a unique morphism  $F_1(A) \rightarrow F_2(A)$  making the following diagram commutative:

$$\begin{array}{ccc}
F_1(P_{i_0}^{\oplus k''_0}) & \xrightarrow{F_1(d''_0)} & F_1(A) \\
F_2(s_0) \circ f_{i_0}^{\oplus k''_0} \downarrow & & \downarrow \\
F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A).
\end{array}$$

Since  $F_2(s_0) \circ f_{i_0}^{\oplus k''_0} = f_{i_0}^{\oplus k_0} \circ F_1(s_0)$ , both  $f_A$  and  $f''_A$  have this property and then they coincide. Similarly one can prove that  $f''_A = f'_A$ .

Therefore  $f_A = f'_A$ , and so  $\tilde{f}(A) := f_A: F_1(A) \rightarrow F_2(A)$  is well defined for every  $A \in \mathbf{F}$ . It is also easy to see that  $\tilde{f}: F_1|_{\mathbf{F}} \rightarrow F_2|_{\mathbf{F}}$  is a natural transformation. Indeed, let  $u: A \rightarrow B$  be a morphism of  $\mathbf{F}$ . Consider a resolution of  $B$

$$\cdots \rightarrow P_{l_j}^{\oplus h_j} \xrightarrow{e_j} P_{l_{j-1}}^{\oplus h_{j-1}} \xrightarrow{e_{j-1}} \cdots \xrightarrow{e_1} P_{l_0}^{\oplus h_0} \xrightarrow{e_0} B \rightarrow 0,$$

where  $l_j \in I$  and  $h_j \in \mathbb{N}$  for every  $j \in \mathbb{N}$ . Reasoning as before, we can find a resolution of  $A$

$$\cdots \rightarrow P_{i_j}^{\oplus k_j} \xrightarrow{d_j} P_{i_{j-1}}^{\oplus k_{j-1}} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} P_{i_0}^{\oplus k_0} \xrightarrow{d_0} A \rightarrow 0$$

and morphisms  $g_j: P_{i_j}^{\oplus k_j} \rightarrow P_{l_j}^{\oplus h_j}$  defining a morphism of complexes compatible with  $u$ . We can now consider the diagram

$$\begin{array}{ccccc}
F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & & \xrightarrow{F_1(d_0)} & F_1(A) \\
\downarrow F_1(g_0) & \searrow f_{i_0}^{\oplus k_0} & & \searrow f_A & \downarrow F_1(u) \\
& & F_2(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_2(d_0)} & F_2(A) \\
& & \downarrow F_2(g_0) & & \downarrow F_2(u) \star \\
& & F_2(P_{l_0}^{\oplus h_0}) & \xrightarrow{F_2(e_0)} & F_2(B) \\
& \nearrow f_{l_0}^{\oplus h_0} & & \nearrow f_B & \\
F_1(P_{l_0}^{\oplus h_0}) & \xrightarrow{F_1(e_0)} & & \xrightarrow{F_1(e_0)} & F_1(B)
\end{array}$$

where all squares but  $\star$  are commutative. Using the same argument as above, we can take  $m > N(A), N(B)$  and truncate the resolutions of  $A$  and  $B$  at step  $m$ . Then, applying (i) and Lemma 3.2, we see that there is a unique morphism  $F_1(A) \rightarrow F_2(B)$  completing the following diagram to a commutative square

$$\begin{array}{ccc}
F_1(P_{i_0}^{\oplus k_0}) & \xrightarrow{F_1(d_0)} & F_1(A) \\
\downarrow F_2(g_0) \circ f_{i_0}^{\oplus k_0} & & \downarrow \\
F_2(P_{l_0}^{\oplus h_0}) & \xrightarrow{F_2(e_0)} & F_2(B).
\end{array}$$

Since  $F_2(g_0) \circ f_{i_0}^{\oplus k_0} = f_{l_0}^{\oplus h_0} \circ F_1(g_0)$ , both  $F_2(u) \circ f_A$  and  $f_B \circ F_1(u)$  have this property. It follows that  $F_2(u) \circ \tilde{f}(A) = \tilde{f}(B) \circ F_1(u)$ , thus proving that  $\tilde{f}: F_1|_{\mathbf{F}} \rightarrow F_2|_{\mathbf{F}}$  is a natural transformation. It is clear by construction that  $\tilde{f}|_{\mathbf{C}} = f$  and that  $\tilde{f}$  is unique with this property.

Since the objects of  $\mathbf{D}_0$  are precisely (up to isomorphism) shifts of objects of  $\mathbf{F}$ , we just need to define  $f_0(A[k])$  for  $A \in \mathbf{F}$  and  $k \in \mathbb{Z}$ . Of course, we must set  $f_0(A[k]) := \tilde{f}(A)[k]$ , and we have just to show that  $f_0(B[k]) \circ F_1(u) = F_2(u) \circ f_0(A)$  for every objects  $A, B \in \mathbf{F}$  and every  $u \in \text{Hom}(A, B[k])$ . Now, there is nothing to prove if  $k < 0$  (because then  $u = 0$ ) or  $k = 0$  (because we have already seen that  $\tilde{f}$  is a natural transformation), so we assume  $k > 0$ . Actually we can reduce to the case  $k = 1$ , thanks to the fact that one can always factor  $u$  as  $u = u_k \circ \dots \circ u_1$ , where (for  $j = 1, \dots, k$ )  $u_j \in \text{Hom}(C_{j-1}[j-1], C_j[j])$  and  $C_0 = A, \dots, C_k = B$  are objects of  $\mathbf{F}$  (see Step 4 in the proof of [24, Prop. B.1]). Now, completing  $u$  to a distinguished triangle  $B \rightarrow C \xrightarrow{v} A \xrightarrow{u} B[1]$ ,  $C$  is again an object of  $\mathbf{F}$ . Then by axiom (TR3) there exists a morphism  $h: F_1(A) \rightarrow F_2(A)$  such that the diagram

$$\begin{array}{ccccccc}
F_1(B) & \longrightarrow & F_1(C) & \xrightarrow{F_1(v)} & F_1(A) & \xrightarrow{F_1(u)} & F_1(B[1]) \\
\downarrow \tilde{f}(B) & & \downarrow \tilde{f}(C) & & \downarrow h & & \downarrow f_0(B[1]) \\
F_2(B) & \longrightarrow & F_2(C) & \xrightarrow{F_2(v)} & F_2(A) & \xrightarrow{F_2(u)} & F_2(B[1])
\end{array}$$

commutes. Since  $\mathrm{Hom}(\mathbf{F}_1(B[1]), \mathbf{F}_2(A)) = 0$  by hypothesis,  $h$  is unique such that  $h \circ \mathbf{F}_1(v) = \mathbf{F}_2(v) \circ \tilde{f}(C)$ . Hence  $h = \tilde{f}(A) = f_0(A)$ , and we conclude that  $f_0(B[1]) \circ \mathbf{F}_1(u) = \mathbf{F}_2(u) \circ f_0(A)$ .  $\square$

Let us specialize to the case of isomorphisms.

**Corollary 3.6.** *With the same hypotheses on  $\mathbf{E}$ ,  $\mathbf{A}$  and  $\mathbf{T}$ , let  $\mathbf{F}_1, \mathbf{F}_2: \mathbf{D}^b(\mathbf{E}) \rightarrow \mathbf{T}$  be exact functors and let  $f: \mathbf{F}_1|_{\mathbf{C}} \xrightarrow{\sim} \mathbf{F}_2|_{\mathbf{C}}$  be an isomorphism of functors. Assume that  $\mathbf{F}_1$  satisfies condition (1) of  $(\Delta)$ . Then there exists a unique isomorphism compatible with shifts  $f_0: \mathbf{F}_1|_{\mathbf{D}_0} \xrightarrow{\sim} \mathbf{F}_2|_{\mathbf{D}_0}$  extending  $f$ .*

*Proof.* Since  $f$  is an isomorphism, we can use Lemma 3.1 in the above argument to show that there is an isomorphism  $\mathbf{F}_1(A) \cong \mathbf{F}_2(A)$ , for all  $A \in \mathbf{D}^b(\mathbf{E}) \cap \mathbf{A}$ . Thus (i) in Proposition 3.5 follows from (1) in  $(\Delta)$ . Therefore, by the above proposition, there is a unique natural transformation compatible with shifts  $f_0: \mathbf{F}_1 \rightarrow \mathbf{F}_2$  extending  $f$ . Uniqueness, applied also to  $f^{-1}$ ,  $f \circ f^{-1}$  and  $f^{-1} \circ f$ , immediately implies that  $f_0$  is an isomorphism  $\square$

**3.3. The criterion: extension to the whole derived category.** In order to extend the natural transformation  $f$  of Proposition 3.5 to  $\mathbf{D}^b(\mathbf{E})$ , the exact functors have to satisfy one more assumption.

**Proposition 3.7.** *Let  $\mathbf{T}$  be a triangulated category and let  $\mathbf{E}$  be a full exact subcategory of a Hom-finite abelian category  $\mathbf{A}$  satisfying (E1), (E2) and (E3). Let  $\mathbf{F}_1, \mathbf{F}_2: \mathbf{D}^b(\mathbf{E}) \rightarrow \mathbf{T}$  be exact functors with a natural transformation  $f: \mathbf{F}_1|_{\mathbf{C}} \rightarrow \mathbf{F}_2|_{\mathbf{C}}$ . Assume moreover the following:*

- (i)  $\mathbf{F}_1$  and  $\mathbf{F}_2$  both satisfy condition (1) of  $(\Delta)$  and

$$\mathrm{Hom}(\mathbf{F}_1(A), \mathbf{F}_2(B)[k]) = 0,$$

for any  $A, B \in \mathbf{D}^b(\mathbf{E}) \cap \mathbf{A}$  and any integer  $k < 0$ ;

- (ii) for all  $C \in \mathbf{D}^b(\mathbf{E})$  with cohomologies in non-positive degrees, there is  $i \in I$  such that

$$\mathrm{Hom}(\mathbf{F}_1(C), \mathbf{F}_2(P_i)) = 0$$

and  $i$  satisfies conditions (1) and (2) of Definition 2.5 for  $H^0(C)$ .

Then there exists a unique natural transformation of exact functors  $g: \mathbf{F}_1 \rightarrow \mathbf{F}_2$  extending  $f$ .

*Proof.* For  $n \in \mathbb{N}$ , denote by  $\mathbf{D}_n$  the (strictly) full subcategory of  $\mathbf{D}^b(\mathbf{E})$  with objects the complexes  $A$  with the following property: there exists  $a \in \mathbb{Z}$  such that  $H^p(A) = 0$  for  $p < a$  or  $p > a + n$ . We are going to prove by induction on  $n$  that  $f$  extends uniquely to a natural transformation compatible with shifts  $f_n: \mathbf{F}_1|_{\mathbf{D}_n} \rightarrow \mathbf{F}_2|_{\mathbf{D}_n}$ . Once we do this, it is obvious that for every object  $A$  of  $\mathbf{D}^b(\mathbf{E})$  we can define  $g(A) := f_n(A)$  if  $A \in \mathbf{D}_n$ , and that  $g$  is then the unique required extension of  $f$ .

The case  $n = 0$  having already been proved in Proposition 3.5, we come to the inductive step from  $n - 1$  to  $n > 0$ . For every object  $A \in \mathbf{D}_n$  we need to define  $f_n(A): \mathbf{F}_1(A) \rightarrow \mathbf{F}_2(A)$ . To this purpose, we can assume that  $H^p(A) = 0$  for  $p < -n$  or  $p > 0$ . If  $A = (\cdots \rightarrow A^0 \xrightarrow{d^0} A^1 \rightarrow \cdots)$ , let  $s: P_i^{\oplus k} \twoheadrightarrow \ker d^0$  (for some  $i \in I$  and  $k \in \mathbb{N}$ ) be an epimorphism such that  $\mathrm{Hom}(\mathbf{F}_1(A), \mathbf{F}_2(P_i)) = 0$ . Notice that  $s$  can be found as follows: after choosing an epimorphism  $P_j^{\oplus l} \twoheadrightarrow \ker d^0$  (with  $j \in I$  and  $l \in \mathbb{N}$ ), take  $i \in I$  which satisfies condition (ii) for  $A \oplus P_j^{\oplus l}$  (so that, in particular, there is an

epimorphism  $P_i^{\oplus k} \twoheadrightarrow P_j^{\oplus l}$ , and define  $s$  to be the composition  $P_i^{\oplus k} \twoheadrightarrow P_j^{\oplus l} \twoheadrightarrow \ker d^0$ . Denoting by  $t: P_i^{\oplus k} \rightarrow A$  the composition of  $s$  with the natural morphism  $\ker d^0 \rightarrow A$ , it is then clear that  $H^0(t): P_i^{\oplus k} \rightarrow H^0(A)$  is an epimorphism. It follows that we have a distinguished triangle

$$(3.6) \quad C[-1] \rightarrow P_i^{\oplus k} \xrightarrow{t} A \xrightarrow{t_1} C$$

with  $C \in \mathbf{D}_{n-1}$ . Hence, by the inductive hypothesis and using axiom (TR3), we obtain a commutative diagram whose rows are distinguished triangles

$$\begin{array}{ccccccc} F_1(C)[-1] & \longrightarrow & F_1(P_i^{\oplus k}) & \xrightarrow{F_1(t)} & F_1(A) & \xrightarrow{F_1(t_1)} & F_1(C) \\ \downarrow f_{n-1}(C)[-1] & & \downarrow f_{n-1}(P_i^{\oplus k}) & & \downarrow f_A & & \downarrow f_{n-1}(C) \\ F_2(C)[-1] & \longrightarrow & F_2(P_i^{\oplus k}) & \xrightarrow{F_2(t)} & F_2(A) & \xrightarrow{F_2(t_1)} & F_2(C) \end{array}$$

for some  $f_A: F_1(A) \rightarrow F_2(A)$ . Observe that, since  $\text{Hom}(F_1(A), F_2(P_i^{\oplus k})) = 0$  by assumption,  $f_A$  is the unique morphism such that the square on the right commutes.

In order to prove that  $f_A$  does not depend on the choice of  $s$ , assume that  $s': P_{i'}^{\oplus k'} \twoheadrightarrow \ker d^0$  is another epimorphism such that  $\text{Hom}(F_1(A), F_2(P_{i'})) = 0$ , and thus inducing another morphism  $f'_A: F_1(A) \rightarrow F_2(A)$ . We claim that we can find a third epimorphism  $s'': P_{i''}^{\oplus k''} \twoheadrightarrow \ker d^0$  such that  $\text{Hom}(F_1(A), F_2(P_{i''})) = 0$  (inducing  $f''_A: F_1(A) \rightarrow F_2(A)$ ) and fitting into a commutative diagram

$$\begin{array}{ccc} P_{i''}^{\oplus k''} & \xrightarrow{w} & P_i^{\oplus k} \\ w' \downarrow & \searrow s'' & \downarrow s \\ P_{i'}^{\oplus k'} & \xrightarrow{s'} & \ker d^0 \end{array}$$

This can be easily seen if one takes  $i'' \in I$  satisfying condition (ii) for  $A \oplus P_j^l$ , where  $j \in I$  and  $l \in \mathbb{N}$  are such that there exists an epimorphism  $P_j^{\oplus l} \twoheadrightarrow P_i^{\oplus k} \times_{\ker d^0} P_{i'}^{\oplus k'}$ . Observing that the morphisms  $t': P_{i'}^{\oplus k'} \rightarrow A$  and  $t'': P_{i''}^{\oplus k''} \rightarrow A$  (induced, respectively, by  $s'$  and  $s''$ ) obviously satisfy  $t \circ w = t'' = t' \circ w'$ , by axiom (TR3) there is a commutative diagram whose rows are distinguished triangles

$$\begin{array}{ccccccc} P_{i''}^{\oplus k''} & \xrightarrow{t''} & A & \xrightarrow{t'_1} & C'' & \longrightarrow & P_{i''}^{\oplus k''}[1] \\ \downarrow w & & \downarrow \text{id} & & \downarrow v & & \downarrow w[1] \\ P_i^{\oplus k} & \xrightarrow{t} & A & \xrightarrow{t_1} & C & \longrightarrow & P_i^{\oplus k}[1] \end{array}$$

for some  $v: C'' \rightarrow C$ . As the diagram

$$\begin{array}{ccccc} F_1(A) & \xrightarrow{F_1(t'_1)} & F_1(C'') & \xrightarrow{F_1(v)} & F_1(C) \\ \downarrow f''_A & & \downarrow f_{n-1}(C'') & & \downarrow f_{n-1}(C) \\ F_2(A) & \xrightarrow{F_2(t'_1)} & F_2(C'') & \xrightarrow{F_2(v)} & F_2(C) \end{array}$$

commutes (the square on the left by definition of  $f''_A$ , the square on the right because  $f_{n-1}$  is a natural transformation by induction) and since  $v \circ t'_1 = t_1$ , we obtain

$$f_{n-1}(C) \circ F_1(t_1) = f_{n-1}(C) \circ F_1(v) \circ F_1(t'_1) = F_2(v) \circ F_2(t'_1) \circ f''_A = F_2(t_1) \circ f''_A.$$

On the other hand,  $f_A$  is the only morphism with the property that  $f_{n-1}(C) \circ F_1(t_1) = F_2(t_1) \circ f_A$ . It follows that  $f_A = f''_A$  and similarly  $f'_A = f''_A$ , thereby proving that  $f_A = f'_A$ . Therefore we can set  $f_n(A) := f_A$ , and more generally  $f_n(A[k]) := f_A[k]$  for every integer  $k$ , thus defining  $f_n$  on every object of  $\mathbf{D}_n$ .

To conclude the inductive step it is enough to show that  $f_n$  is a natural transformation, because then it is clear by construction that  $f_n$  is compatible with shifts, that  $f_n|_{\mathbf{C}} = f_{n-1}|_{\mathbf{C}} = f$  (actually also  $f_n|_{\mathbf{D}_{n-1}} = f_{n-1}$ ) and that  $f_n$  is unique with these properties. So we have to prove that

$$(3.7) \quad f_n(B) \circ F_1(u) = F_2(u) \circ f_n(A)$$

for every morphism  $u: A \rightarrow B$  of  $\mathbf{D}_n$ . Recall that in  $D^b(\mathbf{E})$  we can write  $u = v \circ w^{-1}$ , where  $v$  and  $w$  are (represented by) morphisms of complexes and  $w$  is a quasi-isomorphism (hence  $v$  and  $w$  are again in  $\mathbf{D}_n$ ). Thus  $f_n$  is compatible with  $u$  (namely, (3.7) holds) if it is compatible both with  $w^{-1}$  (or, equivalently, with  $w$ ) and with  $v$ . In other words, it is harmless to assume directly that  $u$  is a morphism of complexes, denoted by  $A = (\cdots \rightarrow A^0 \xrightarrow{d^0} A^1 \rightarrow \cdots)$  and  $B = (\cdots \rightarrow B^0 \xrightarrow{e^0} B^1 \rightarrow \cdots)$ . We can also assume that, as before,  $H^p(A) = 0$  for  $p < -n$  or  $p > 0$ . Moreover, we denote by  $c$  the greatest integer such that  $H^c(B) \neq 0$  (of course, if  $B \cong 0$  there is nothing to prove). Now our aim is to show that the problem of verifying (3.7) can be reduced to a similar problem with another “simpler” morphism in place of  $u$ . To this purpose we distinguish two cases according to the value of  $c$ .

If  $c < 0$ , choose  $j \in I$  which satisfies parts (1) and (2) of Definition 2.5 for  $\ker d^0 \oplus \ker e^{-1}$ , let  $P_j^{\oplus l} \rightarrow \ker d^0 \oplus \ker e^{-1}$  be an epimorphism, and take  $i \in I$  satisfying condition (ii) for  $A \oplus P_j^{\oplus l}$ . Then, reasoning as before, we get an epimorphism  $s: P_i^{\oplus k} \rightarrow \ker d^0$  which can be used to define  $f_A$ . Moreover, denoting by  $t: P_i^{\oplus k} \rightarrow A$  the morphism (of complexes) induced by  $s$ , we claim that  $u \circ t$  is homotopic to 0. Indeed,  $u \circ t$  is given by  $w \circ s$  for some  $w: \ker d^0 \rightarrow \ker e^0 \subseteq B^0$  in  $\mathbf{A}$ . Since  $\ker e^0 = \text{im } e^{-1}$  (because  $c < 0$ ), there is a short exact sequence  $0 \rightarrow \ker e^{-1} \rightarrow B^{-1} \rightarrow \ker e^0 \rightarrow 0$  in  $\mathbf{A}$ . Then there exists  $w': P_i^{\oplus k} \rightarrow B^{-1}$  such that  $w \circ s = e^{-1} \circ w'$ , because the composition  $P_i^{\oplus k} \xrightarrow{w \circ s} \ker e^0 \rightarrow \ker e^{-1}[1]$  is 0 by assumption. This proves that  $u \circ t$  is homotopic to 0, whence it is 0 also in  $D^b(\mathbf{E})$ . From this and from the distinguished triangle (3.6) it follows that  $u = v \circ t_1$  for some  $v: C \rightarrow B$  (with  $C \in \mathbf{D}_{n-1}$ ). As  $f_n$  is compatible with  $t_1$  by definition of  $f_A = f_n(A)$ , in order to check (3.7) it is therefore enough to show that  $f_n$  is compatible with  $v$ . Notice that, if  $A \in \mathbf{D}_m$  for some  $0 < m \leq n$ , then  $C \in \mathbf{D}_{m-1}$ . On the other hand, if  $A \in \mathbf{D}_0$  (hence  $A$  is isomorphic to an object of  $\mathbf{F}$ ), then  $C \in \mathbf{D}_0$  and  $C[-1]$  is isomorphic to an object of  $\mathbf{F}$ . So in this last case, passing from  $u$  to  $v[-1]$ ,  $c$  increases by 1.

If  $c \geq 0$ , choose an epimorphism  $P_j^{\oplus l} \rightarrow \ker e^c$  (with  $j \in I$  and  $l \in \mathbb{N}$ ) and take  $i \in I$  satisfying condition (ii) for  $A[c] \oplus B[c] \oplus P_j^{\oplus l}$ . Then, as usual, we can find an epimorphism  $s': P_i^{\oplus k} \rightarrow \ker e^c$  which can be used to define  $f_{B[c]}$ . Denoting by  $t': P_i^{\oplus k}[-c] \rightarrow B$  the morphism induced by  $s'$ , and extending it to a distinguished triangle

$$C'[-1] \rightarrow P_i^{\oplus k}[-c] \xrightarrow{t'} B \xrightarrow{t'_1} C'$$

(with  $C' \in \mathbf{D}_{n-1}$ ), we claim that (3.7) follows once one proves that  $f_n$  is compatible with  $v' := t'_1 \circ u: A \rightarrow C'$ . To see this, observe that in the diagram

$$\begin{array}{ccccc} \mathbf{F}_1(A) & \xrightarrow{\mathbf{F}_1(u)} & \mathbf{F}_1(B) & \xrightarrow{\mathbf{F}_1(t'_1)} & \mathbf{F}_1(C') \\ \downarrow f_n(A) & & \downarrow f_n(B) & & \downarrow f_n(C') \\ \mathbf{F}_2(A) & \xrightarrow{\mathbf{F}_2(u)} & \mathbf{F}_2(B) & \xrightarrow{\mathbf{F}_2(t'_1)} & \mathbf{F}_2(C') \end{array}$$

the square on the right commutes by definition of  $f_{B[c]}[-c] = f_n(B)$ , whence (assuming compatibility of  $f_n$  with  $v'$ )

$$\mathbf{F}_2(t'_1) \circ (f_n(B) \circ \mathbf{F}_1(u) - \mathbf{F}_2(u) \circ f_n(A)) = f_n(C') \circ \mathbf{F}_1(v') - \mathbf{F}_2(v') \circ f_n(A) = 0.$$

It follows that  $f_n(B) \circ \mathbf{F}_1(u) - \mathbf{F}_2(u) \circ f_n(A)$  factors through  $\mathbf{F}_2(t')$ , and then it must be 0 (which means that (3.7) holds) because  $\mathrm{Hom}(\mathbf{F}_1(A), \mathbf{F}_2(P_i^{\oplus k}[-c])) = 0$  by the choice of  $i$ . Observe that, similarly as above, if  $B \in \mathbf{D}_m$  for some  $0 < m \leq n$ , then  $C' \in \mathbf{D}_{m-1}$ , whereas, if  $B \in \mathbf{D}_0$ , then  $C' \in \mathbf{D}_0$  and, passing from  $u$  to  $v'$ ,  $c$  decreases by 1.

To finish the proof, just note that, if one repeats the above procedure a sufficient number of times, then one necessarily encounters both cases ( $c < 0$  and  $c \geq 0$ ), thus reducing to check compatibility of  $f_n$  with a morphism of  $\mathbf{D}_{n-1}$ , when it holds by induction.  $\square$

In the paper we will need the following special case of the above result.

**Corollary 3.8.** *With the same hypotheses on  $\mathbf{E}$ ,  $\mathbf{A}$  and  $\mathbf{T}$ , let  $\mathbf{F}_1, \mathbf{F}_2: \mathbf{D}^b(\mathbf{E}) \rightarrow \mathbf{T}$  be exact functors and let  $f: \mathbf{F}_1|_{\mathbf{C}} \xrightarrow{\sim} \mathbf{F}_2|_{\mathbf{C}}$  be an isomorphism. Assume moreover that  $\mathbf{F}_1$  satisfies  $(\Delta)$ . Then there exists a unique isomorphism of exact functors  $g: \mathbf{F}_1 \xrightarrow{\sim} \mathbf{F}_2$  extending  $f$ .*

*Proof.* As  $f$  is an isomorphism, we can apply Corollary 3.6 so that  $\mathbf{F}_1(A) \cong \mathbf{F}_2(A)$ , for all  $A \in \mathbf{D}^b(\mathbf{E}) \cap \mathbf{A}$ . Hence hypothesis (i) in Proposition 3.7 follows from  $(\Delta)$ . Analogously, for (ii) we use that  $\mathbf{F}_1(P_i) \cong \mathbf{F}_2(P_i)$  by assumption. Thus Proposition 3.7 applies and we get a unique natural transformation of exact functors  $g: \mathbf{F}_1 \rightarrow \mathbf{F}_2$  extending  $f$ . The fact that  $g$  is an isomorphism is again a formal consequence of uniqueness, as in the proof of Corollary 3.6.  $\square$

In the case  $\mathbf{E} = \mathbf{A}$ , we are going to give a sufficient condition under which  $(\Delta)$  is automatically satisfied. We leave it to the reader to formulate a similar statement which ensures that the hypotheses of Proposition 3.7 are satisfied.

**Lemma 3.9.** *Let  $\mathbf{F}: \mathbf{D}^b(\mathbf{A}) \rightarrow \mathbf{T}$  be an exact functor admitting a left adoint and satisfying  $(\diamond)$ . Assume moreover that  $\{P_i\}_{i \in I}$  is a weakly ample set in  $\mathbf{A}$ . Then  $\mathbf{F}$  satisfies  $(\Delta)$  as well.*

*Proof.* Observing that part (1) of  $(\Delta)$  coincides with  $(\diamond)$  because  $\mathbf{E} = \mathbf{A}$ , it remains to prove part (2) of  $(\Delta)$ . Denoting by  $\mathbf{F}^*: \mathbf{T} \rightarrow \mathbf{D}^b(\mathbf{A})$  the left adjoint of  $\mathbf{F}$ , we claim that  $H^p(\mathbf{F}^* \circ \mathbf{F}(A)) = 0$  for any  $A \in \mathbf{A}$  and for any  $p > 0$ . Indeed, otherwise there would exist  $A \in \mathbf{A}$  and  $m > 0$  with a non-zero morphism  $\mathbf{F}^* \circ \mathbf{F}(A) \rightarrow H^m(\mathbf{F}^* \circ \mathbf{F}(A))[-m]$  (it is enough to let  $m$  be the largest integer such that  $H^m(\mathbf{F}^* \circ \mathbf{F}(A)) \neq 0$ ). But then

$$0 \neq \mathrm{Hom}(\mathbf{F}^* \circ \mathbf{F}(A), H^m(\mathbf{F}^* \circ \mathbf{F}(A))[-m]) \cong \mathrm{Hom}(\mathbf{F}(A), \mathbf{F}(H^m(\mathbf{F}^* \circ \mathbf{F}(A))[-m]))$$

by adjunction, contradicting  $(\diamond)$ . Clearly the above implies more generally that  $H^p(\mathbf{F}^* \circ \mathbf{F}(C)) = 0$  for any  $C \in \mathbf{D}^b(\mathbf{A})$  having cohomologies in non-positive degrees. Then for such an object  $C$  and for any  $i \in I$  we have

$$\mathrm{Hom}(\mathbf{F}(C), \mathbf{F}(P_i)) \cong \mathrm{Hom}(\mathbf{F}^* \circ \mathbf{F}(C), P_i) \cong \mathrm{Hom}(H^0(\mathbf{F}^* \circ \mathbf{F}(C)), P_i).$$

Therefore part (2) of  $(\Delta)$  is satisfied if one takes  $i \in I$  as in Definition 2.5 for  $H^0(\mathbf{F}^* \circ \mathbf{F}(C)) \oplus H^0(C)$ .  $\square$

Combining the above result with Corollary 3.8 immediately gives a proof of Proposition 3.3.

**3.4. The geometric case and some examples.** In this section we want to clarify which abelian category  $\mathbf{A}$  and exact subcategory  $\mathbf{E}$  have to be taken in order to use the results in Section 3.3 to prove Theorems 1.1 and 1.2.

Therefore let  $X$  be a quasi-projective scheme and let  $Z$  be a projective subscheme of  $X$ . Assume further that  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$  for all  $i > 0$ . Set

$$\mathbf{A} := \mathbf{Coh}_Z(X) \quad \mathbf{E} := \mathbf{Perf}_Z(X) \cap \mathbf{Coh}_Z(X).$$

**Proposition 3.10.** *Under the above assumptions,  $\mathbf{E}$  is a full exact subcategory of  $\mathbf{A}$ , (E1)–(E3) are satisfied and  $\mathbf{Perf}_Z(X) = \mathbf{D}^b(\mathbf{E}) \subseteq \mathbf{D}^b(\mathbf{A})$ .*

*Proof.* The subcategory  $\mathbf{E}$  is closed under extensions, hence  $\mathbf{E}$  is a full exact subcategory of  $\mathbf{A}$  (see [20, Sect. 4]). Condition (E1) follows from the fact that, if  $f$  is an admissible epimorphism in  $\mathbf{E}$ , then  $\ker f \in \mathbf{E}$ . As  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$  for all  $i > 0$ , (E2) holds true taking  $\{P_i\}_{i \in I} = \mathbf{Amp}(Z, X, H)$  defined in (2.3) (with  $H$  an ample divisor on  $X$ ).

Obviously  $\mathbf{D}^b(\mathbf{E})$  is a full subcategory of  $\mathbf{Perf}_Z(X)$ . To show that they are actually equal, one has to apply an induction argument similar to the one in the first part of the proof of Proposition 3.7. To give a hint, let  $\mathbf{D}_n$  be the (strictly) full subcategory of  $\mathbf{Perf}_Z(X)$  with objects the complexes  $\mathcal{A}$  with the following property: there exists  $a \in \mathbb{Z}$  such that  $H^p(\mathcal{A}) = 0$  for  $p < a$  or  $p > a + n$ . Given  $\mathcal{A} \in \mathbf{Perf}_Z(X)$ , there exists  $n \geq 0$  such that  $\mathcal{A} \in \mathbf{D}_n$ , and one can prove that  $\mathcal{A} \in \mathbf{D}^b(\mathbf{E})$  by induction on  $n$ . Indeed, if  $n \leq 1$ , there is nothing to prove. Otherwise we can assume without loss of generality that  $H^p(\mathcal{A}) = 0$  for  $p < -n$  or  $p > 0$ . Then  $\mathcal{A}$  sits in a distinguished triangle

$$\mathcal{C}[-1] \rightarrow P_i^{\oplus k} \rightarrow \mathcal{A} \rightarrow \mathcal{C},$$

where  $P_i \in \mathbf{Amp}(Z, X, H)$ ,  $k \in \mathbb{N}$  and  $\mathcal{C} \in \mathbf{D}_{n-1}$ .

As for (E3), one can prove more generally that for every  $\mathcal{A} \in \mathbf{D}^b(\mathbf{E}) = \mathbf{Perf}_Z(X)$  there exists an integer  $N(\mathcal{A})$  such that  $\mathrm{Hom}_{\mathbf{D}^b(\mathbf{A})}(\mathcal{A}, \mathcal{B}[i]) = 0$ , for every  $i > N(\mathcal{A})$  and every  $\mathcal{B} \in \mathbf{A}$ . Indeed, this follows from the isomorphism

$$\mathrm{Hom}_{\mathbf{D}^b(\mathbf{A})}(\mathcal{A}, \mathcal{B}[i]) \cong \mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{O}_X, \mathcal{A}^\vee \otimes \mathcal{B}[i]),$$

which holds because  $\mathcal{A}$  is perfect.  $\square$

**Remark 3.11.** In view of Proposition 2.8, it is easy to see that, if  $X$ ,  $Z$ ,  $\mathbf{E}$  and  $\mathbf{A}$  are as above, then condition  $(*)$  in the introduction implies  $(\Delta)$ . Therefore, in the proof of Theorems 1.1 and 1.2 we can freely use the results in Section 3.3.

It may be useful to keep in mind some examples of exact functors satisfying  $(*)$ .

**Example 3.12.** In this example we assume that  $X_1$  is a quasi-projective scheme with a projective subscheme  $Z_1$  such that  $\mathcal{O}_{iZ_1} \in \mathbf{Perf}(X_1)$ , for all  $i > 0$ , and  $T_0(\mathcal{O}_{Z_1}) = 0$ .

(i) Using (3) in Definition 2.5, it is very easy to verify that full functors  $\mathbf{F}: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  satisfy  $(*)$  for any scheme  $X_2$  containing a subscheme  $Z_2$  proper over  $\mathbb{k}$ .

(ii) For the same reason, a trivial example of a functor with the property  $(*)$  but which is not full is  $\mathrm{id} \oplus \mathrm{id}: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_1}(X_1)$ .

(iii) Following the same argument as in [11], in the supported setting one may take exact functors  $D_{Z_1}^b(X_1) \rightarrow D_{Z_2}^b(X_2)$  induced by exact full functors  $\mathbf{Coh}_{Z_1}(X_1) \rightarrow \mathbf{Coh}_{Z_2}(X_2)$ , where  $X_1$  and  $X_2$  are smooth quasi-projective varieties. These functors obviously satisfy  $(*)$ .

We conclude this section with the following easy result making clear that in the non-supported smooth case,  $(\diamond)$  is equivalent to  $(*)$  in the introduction.

**Proposition 3.13.** *Let  $X_1$  be a smooth projective scheme such that  $\dim(X_1) > 0$  and let  $X_2$  be a scheme containing a subscheme  $Z_2$  which is proper over  $\mathbb{k}$ . Then an exact functor  $\mathbf{F}: D^b(X_1) \rightarrow D_{Z_2}^b(X_2)$  satisfies  $(*)$  if and only if it satisfies  $(\diamond)$ .*

*Proof.* Clearly it is enough to show that  $(\diamond)$  implies (2) in  $(*)$ . Since  $Z_1 = X_1$ ,  $\mathcal{O}_{|iZ_1}(jH_1) = \mathcal{O}_{X_1}(jH_1)$  for all  $i, j \in \mathbb{Z}$ . Hence it is enough to show that for any  $\mathcal{A} \in D^b(X_1)$  with trivial cohomologies in (strictly) positive degrees, there is  $N \in \mathbb{Z}$  such that  $\mathrm{Hom}(\mathbf{F}(\mathcal{A}), \mathbf{F}(\mathcal{O}_{X_1}(iH_1))) = 0$  for any  $i < N$ . Observing that  $\mathbf{F}$  has a left adjoint by [5] (see also [11, Rmk. 2.1]), Lemma 3.9 implies that there exists at least one such  $i$ . The full thesis can be proved in a similar way, using the last statement of Proposition 2.8.  $\square$

**Example 3.14.** In view of Proposition 3.13 and of [11, Prop. 5.1], a non-trivial class of exact functors satisfying  $(*)$  is provided by the functors  $D^b(X_1) \rightarrow D^b(X_2)$  induced by exact functors  $\mathbf{Coh}(X_1) \rightarrow \mathbf{Coh}(X_2)$ . Here we assume that  $X_1$  and  $X_2$  are smooth projective varieties and that  $\dim(X_1) > 0$ .

As a consequence of Proposition 3.13, Theorem 1.1 generalizes the main result of [11] when the twists from the Brauer groups are trivial.

#### 4. ENHANCEMENTS AND EXISTENCE OF FOURIER–MUKAI KERNELS

In this section we show how to construct Fourier–Mukai kernels for functors satisfying the condition  $(*)$  defined in the introduction. This extends several results already present in the literature. Moreover we show that, in the supported setting, the Fourier–Mukai kernels have to be quasi-coherent rather than coherent. We need also to recall some basic facts about dg-categories. As an application of this machinery and of the results in the previous sections, we get the proof of Theorem 1.2.

**4.1. Dg-categories.** In this section we give a quick introduction to some basic definitions and results about dg-categories and dg-functors. For a survey on the subject, the reader can have a look at [21].

Recall that a *dg-category* is an additive category  $\mathbf{A}$  such that, for all  $A, B \in \mathrm{Ob}(\mathbf{A})$ , the morphism spaces  $\mathrm{Hom}(A, B)$  are  $\mathbb{Z}$ -graded  $\mathbb{k}$ -modules with a differential  $d: \mathrm{Hom}(A, B) \rightarrow \mathrm{Hom}(A, B)$  of degree 1 compatible with the composition.

**Example 4.1.** (i) Any additive category  $\mathbf{A}$  has a (trivial) structure of dg-category.

(ii) For a dg-category  $\mathbf{A}$ , one defines the opposite dg-category  $\mathbf{A}^\circ$  with  $\text{Ob}(\mathbf{A}^\circ) = \text{Ob}(\mathbf{A})$  while  $\text{Hom}_{\mathbf{A}^\circ}(A, B) := \text{Hom}_{\mathbf{A}}(B, A)$ .

(iii) For  $X$  a quasi-compact quasi-separated scheme containing a subscheme  $Z$  proper over  $\mathbb{k}$ , we denote by  $C_Z^{\text{dg}}(X)$  the dg-category of unbounded complexes of objects in  $\mathbf{Qcoh}_Z(X)$  and by  $\text{Ac}_Z^{\text{dg}}(X)$  the dg-subcategory of  $C_Z^{\text{dg}}(X)$  consisting of acyclic complexes. Following [13], one can form the quotient  $D_Z^{\text{dg}}(X) := C_Z^{\text{dg}}(X)/\text{Ac}_Z^{\text{dg}}(X)$  which is again a dg-category.

Given a dg-category  $\mathbf{A}$  we denote by  $H^0(\mathbf{A})$  its *homotopy* category. The objects of  $H^0(\mathbf{A})$  are the same as those of  $\mathbf{A}$  while the morphisms are obtained by taking the 0-th cohomology  $H^0(\text{Hom}_{\mathbf{A}}(A, B))$  of the complex  $\text{Hom}_{\mathbf{A}}(A, B)$ . If  $\mathbf{A}$  is pre-triangulated (see [21] for the definition), then  $H^0(\mathbf{A})$  has a natural structure of triangulated category.

**Example 4.2.** If  $X$  and  $Z$  are as in Example 4.1, the dg-category  $D_Z^{\text{dg}}(X)$  is pre-triangulated and  $H^0(D_Z^{\text{dg}}(X)) \cong D_Z(\mathbf{Qcoh}(X))$ . Notice that the supported case is entirely analogous to the non-supported one, for which one can have a look at [21, 27].

A *dg-functor*  $F: \mathbf{A} \rightarrow \mathbf{B}$  is the datum of a map  $\text{Ob}(\mathbf{A}) \rightarrow \text{Ob}(\mathbf{B})$  and of morphisms of dg  $\mathbb{k}$ -modules  $\text{Hom}_{\mathbf{A}}(A, B) \rightarrow \text{Hom}_{\mathbf{B}}(F(A), F(B))$ , for  $A, B \in \text{Ob}(\mathbf{A})$ , which are compatible with the composition and the units.

For a small dg-category  $\mathbf{A}$ , one can consider the pre-triangulated dg-category  $\text{Mod-}\mathbf{A}$  of *right dg  $\mathbf{A}$ -modules*. A right dg  $\mathbf{A}$ -module is a dg-functor  $M: \mathbf{A}^\circ \rightarrow \text{Mod-}\mathbb{k}$ , where  $\text{Mod-}\mathbb{k}$  is the dg-category of dg  $\mathbb{k}$ -modules. The full dg-subcategory of acyclic right dg-modules is denoted by  $\text{Ac}(\mathbf{A})$ , and  $H^0(\text{Ac}(\mathbf{A}))$  is a full triangulated subcategory of the homotopy category  $H^0(\text{Mod-}\mathbf{A})$ . Hence the *derived category* of the dg-category  $\mathbf{A}$  is the Verdier quotient

$$D^{\text{dg}}(\mathbf{A}) := H^0(\text{Mod-}\mathbf{A})/H^0(\text{Ac}(\mathbf{A})).$$

A right dg  $\mathbf{A}$ -module is *representable* if it is contained in the image of the Yoneda functor

$$h: \mathbf{A} \rightarrow \text{Mod-}\mathbf{A} \quad A \mapsto \text{Hom}_{\mathbf{A}}(-, A) =: h^A.$$

A right dg  $\mathbf{A}$ -module is *free* if it is isomorphic to a direct sum of dg-modules of the form  $h^A[m]$ , where  $A \in \mathbf{A}$  and  $m \in \mathbb{Z}$ . A right dg  $\mathbf{A}$ -module  $M$  is *semi-free* if it has a filtration

$$0 = \Phi_0 \subseteq \Phi_1 \subseteq \dots = M$$

such that  $\Phi_i/\Phi_{i-1}$  is free, for all  $i$ . We denote by  $\text{SF}(\mathbf{A})$  the full dg-subcategory of semi-free dg-modules, while  $\text{SF}_{\text{fg}}(\mathbf{A}) \subseteq \text{SF}(\mathbf{A})$  is the full dg-subcategory of *finitely generated semi-free* dg-modules. Namely, there is  $n$  such that  $\Phi_n = M$  and each  $\Phi_i/\Phi_{i-1}$  is a finite direct sum of dg-modules of the form  $h^A[m]$ . The dg-modules which are homotopy equivalent to direct summands of finitely generated semi-free dg-modules are called *perfect* and they form a full dg-subcategory  $\text{Perf}^{\text{dg}}(\mathbf{A})$ .

Following [21, 27], given two dg-categories  $\mathbf{A}$  and  $\mathbf{B}$ , we denote by  $\text{rep}(\mathbf{A}, \mathbf{B})$  the full subcategory of the derived category  $D^{\text{dg}}(\mathbf{A}^\circ \otimes \mathbf{B})$  of  $\mathbf{A}$ - $\mathbf{B}$ -bimodules  $C$  such that the functor  $(-)\otimes_{\mathbf{A}} C: D^{\text{dg}}(\mathbf{A}) \rightarrow D^{\text{dg}}(\mathbf{B})$  sends the representable  $\mathbf{A}$ -modules to objects which are isomorphic to representable  $\mathbf{B}$ -modules. A *quasi-functor* is an object in  $\text{rep}(\mathbf{A}, \mathbf{B})$  which is represented by a dg-functor  $\mathbf{A} \rightarrow$

$\text{Mod-}\mathbf{B}$  whose essential image consists of dg  $\mathbf{B}$ -modules quasi-isomorphic to representable  $\mathbf{B}$ -modules. Notice that a quasi-functor  $\mathbf{M} \in \text{rep}(\mathbf{A}, \mathbf{B})$  defines a functor  $H^0(\mathbf{M}): H^0(\mathbf{A}) \rightarrow H^0(\mathbf{B})$ .

Given two pre-triangulated dg-categories  $\mathbf{A}$  and  $\mathbf{B}$  and an exact functor  $F: H^0(\mathbf{A}) \rightarrow H^0(\mathbf{B})$ , a *dg-lift* of  $F$  is a quasi-functor  $\mathbf{G} \in \text{rep}(\mathbf{A}, \mathbf{B})$  such that  $H^0(\mathbf{G}) \cong F$ .

An *enhancement* of a triangulated category  $\mathbf{T}$  is a pair  $(\mathbf{A}, \alpha)$ , where  $\mathbf{A}$  is a pre-triangulated dg-category and  $\alpha: H^0(\mathbf{A}) \rightarrow \mathbf{T}$  is an exact equivalence. The enhancement  $(\mathbf{A}, \alpha)$  of  $\mathbf{T}$  is *unique* if for any enhancement  $(\mathbf{B}, \beta)$  of  $\mathbf{T}$  there exists a quasi-functor  $\gamma: \mathbf{A} \rightarrow \mathbf{B}$  such that  $H^0(\gamma): H^0(\mathbf{A}) \rightarrow H^0(\mathbf{B})$  is an exact equivalence. We say that the enhancement is *strongly unique* if moreover  $\alpha \cong \beta \circ H^0(\gamma)$ .

**Example 4.3.** Let  $X$  and  $Z$  be again as in Example 4.1. The dg-category  $D_Z^{\text{dg}}(X)$  is an enhancement of  $D_Z(\mathbf{Qcoh}(X))$ .

**4.2. Enhancements and the proof of Theorem 1.2.** Let  $X$  be a quasi-projective scheme containing a projective subscheme  $Z$  and let  $H$  be an ample divisor on  $X$ . Assume that  $\mathcal{O}_{iZ} \in \mathbf{Perf}(X)$  for all  $i > 0$  (hence the full subcategory  $\mathbf{A} := \mathbf{Amp}(Z, X, H)$  defined in (2.3) is contained in  $\mathbf{Perf}_Z(X)$ ). Considered as a dg-category,  $\mathbf{A}$  gives rise to an enhancement of  $D_Z(\mathbf{Qcoh}(X))$ . More precisely, as a straightforward consequence of the results in [24], we have the following result which is used in the rest of the section.

**Lemma 4.4.** *There exists an exact equivalence  $\varphi: D_Z(\mathbf{Qcoh}(X)) \rightarrow D^{\text{dg}}(\mathbf{A})/\mathbf{L}$ , for some localizing subcategory  $\mathbf{L} \subseteq D^{\text{dg}}(\mathbf{A})$ , such that, for any  $P \in \mathbf{A}$ , we have  $\varphi^{-1}(\pi(h^P)) = P$ , where  $\pi: D^{\text{dg}}(\mathbf{A}) \rightarrow D^{\text{dg}}(\mathbf{A})/\mathbf{L}$  is the localizing functor and  $\varphi^{-1}$  is a quasi-inverse of  $\varphi$ . Moreover  $D_Z(\mathbf{Qcoh}(X))$  has a unique enhancement.*

*Proof.* By Proposition 2.8, the category  $\mathbf{Amp}(Z, X, H)$  is a set of compact generators for the Grothendieck category  $\mathbf{Qcoh}_Z(X)$ . Thus, the first assertion follows from the main result in [12] and [24, Lemma 7.2] because  $D_Z(\mathbf{Qcoh}(X)) \cong D(\mathbf{Qcoh}_Z(X))$  (see Proposition 2.3). The second part of the statement is a straightforward consequence of [24, Thm. 7.5].  $\square$

Now we want to prove Theorem 1.2 and so we assume further that  $T_0(\mathcal{O}_Z) = 0$ . By Proposition 3.10, in our setting,  $\mathbf{Perf}_Z(X) \cong D^b(\mathbf{E})$ , for an exact category  $\mathbf{E}$ . We can show an interesting property of the localizing subcategory  $\mathbf{L}$  in the statement of Lemma 4.4.

**Lemma 4.5.** *Under the above assumptions,  $\mathbf{L}^c = \mathbf{L} \cap D^{\text{dg}}(\mathbf{A})^c$ .*

*Proof.* As in the proof of Lemma 4.4 and according to the discussion in Section 7 of [24], we have that  $\pi(h^P) \in (D^{\text{dg}}(\mathbf{A})/\mathbf{L})^c$ , for all  $P \in \mathbf{A}$ . Thus the right adjoint  $\omega: D^{\text{dg}}(\mathbf{A})/\mathbf{L} \rightarrow D^{\text{dg}}(\mathbf{A})$  of  $\pi$  (which exists in view, for example, of [12]) preserves arbitrary direct sums.

Now, to get the desired conclusion, we just need to prove that  $\mathbf{L}^c \subseteq D^{\text{dg}}(\mathbf{A})^c$  and so that, given a collection  $\{X_i\}_{i \in I}$  of objects in  $D^{\text{dg}}(\mathbf{A})$  such that  $\bigoplus_{i \in I} X_i$  exists in  $D^{\text{dg}}(\mathbf{A})$ , then the natural map  $\bigoplus_i \text{Hom}(L, X_i) \rightarrow \text{Hom}(L, \bigoplus_i X_i)$  is an isomorphism, for all  $L \in \mathbf{L}^c$ .

Observe that, for all  $i \in I$ , we have a distinguished triangle

$$(4.1) \quad L_i \longrightarrow X_i \longrightarrow \omega \circ \pi(X_i),$$

where  $\pi(L_i) \cong 0$ . Applying the derived functor  $\mathbf{R}\mathrm{Hom}(L, -)$  to it yields a distinguished triangle

$$\mathbf{R}\mathrm{Hom}(L, L_i) \longrightarrow \mathbf{R}\mathrm{Hom}(L, X_i) \longrightarrow \mathbf{R}\mathrm{Hom}(L, \omega \circ \pi(X_i)).$$

But now  $\mathbf{R}\mathrm{Hom}(L, \omega \circ \pi(X_i)) \cong \mathbf{R}\mathrm{Hom}(\pi(L), \pi(X_i)) = 0$ , as  $\pi(L) \cong 0$ . Hence we have an isomorphism  $f_i: \mathrm{Hom}(L, L_i) \xrightarrow{\sim} \mathrm{Hom}(L, X_i)$ .

Taking direct sums of (4.1) and applying again  $\mathbf{R}\mathrm{Hom}(L, -)$ , we get a distinguished triangle

$$(4.2) \quad \mathbf{R}\mathrm{Hom}(L, \oplus_i L_i) \longrightarrow \mathbf{R}\mathrm{Hom}(L, \oplus_i X_i) \longrightarrow \mathbf{R}\mathrm{Hom}(L, \oplus_i \omega \circ \pi(X_i)).$$

As  $\omega$  commutes with direct sums, the same argument as above shows that there is an isomorphism  $f: \mathrm{Hom}(L, \oplus_i L_i) \xrightarrow{\sim} \mathrm{Hom}(L, \oplus_i X_i)$ .

Defining  $f' := \oplus_i f_i: \oplus_i \mathrm{Hom}(L, L_i) \xrightarrow{\sim} \oplus_i \mathrm{Hom}(L, X_i)$ , we have a commutative diagram

$$\begin{array}{ccc} \oplus_i \mathrm{Hom}(L, L_i) & \xrightarrow{f'} & \oplus_i \mathrm{Hom}(L, X_i) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(L, \oplus_i L_i) & \xrightarrow{f} & \mathrm{Hom}(L, \oplus_i X_i). \end{array}$$

As the left vertical arrow is an isomorphism (being  $L$  a compact object in  $\mathbf{L}$ ), the right vertical one is an isomorphism as well. This concludes the proof.  $\square$

As a consequence of Lemma 4.4, there is an equivalence  $\alpha: \mathbf{D}^b(\mathbf{E}) \rightarrow (\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c$  induced by  $\varphi$  in Lemma 4.4. Let now  $(\mathbf{B}, \beta)$  be an enhancement of  $\mathbf{D}^b(\mathbf{E})$ . Set  $\gamma := \alpha \circ \beta: H^0(\mathbf{B}) \rightarrow (\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c$ . Set  $\mathbf{C}$  to be an enhancement of  $(\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c$  as in [24, Thm. 6.4]. Namely,  $\mathbf{C} \subseteq \mathrm{SF}(\mathbf{A})/(\mathbf{L} \cap \mathrm{SF}(\mathbf{A}))$  consists of the compact objects in  $\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L}$ . Here we use the same letter  $\mathbf{L}$  to denote the full dg-subcategory of  $\mathcal{M}od\text{-}\mathbf{A}$  with the same objects as  $\mathbf{L} \subseteq \mathbf{D}^{\mathrm{dg}}(\mathbf{A})$ . Therefore  $H^0(\mathbf{C})$  naturally identifies with  $(\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c$ .

Due to Lemma 4.5, [24, Thm. 6.4] applies providing a quasi-equivalence  $\delta: \mathbf{C} \rightarrow \mathbf{B}$  together with an isomorphism of functors

$$H^0(\delta) \circ \pi \circ h^{(-)} \xrightarrow{\sim} \gamma^{-1} \circ \pi \circ h^{(-)}.$$

By Lemma 4.4, there is an isomorphism of functors  $\alpha|_{\mathbf{A}} \cong \pi \circ h^{(-)}|_{\mathbf{A}}$ . Thus, if we set  $\mathbf{F}_1 := \gamma^{-1} \circ \alpha$  and  $\mathbf{F}_2 := H^0(\delta) \circ \alpha$ , there exists an isomorphism of functors

$$(4.3) \quad \mathbf{F}_1|_{\mathbf{A}} \xrightarrow{\sim} \mathbf{F}_2|_{\mathbf{A}}.$$

By Corollary 3.8, it extends to a unique isomorphism  $\mathbf{F}_1 \cong \mathbf{F}_2$  and so  $\gamma^{-1}$  and  $H^0(\delta)$  are isomorphic. Notice that this is the point where we use that  $T_0(\mathcal{O}_Z) = 0$  as, under this assumption, a full functor such that  $T_0(\mathcal{O}_Z) = 0$  certainly satisfies  $(\Delta)$  and  $(*)$  (see Example 3.12).

This proves that the enhancement of  $\mathbf{Perf}_Z(X)$  is strongly unique as stated in Theorem 1.2. Indeed, suppose that  $(\mathbf{B}_1, \beta_1)$  and  $(\mathbf{B}_2, \beta_2)$  are enhancements of  $(\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c$ . By the above discussion, there are quasi-equivalences  $\delta_i: \mathbf{B}_i \rightarrow \mathbf{C}$  and unique isomorphisms  $H^0(\delta_i) \cong \beta_i$ . To conclude, if we set  $\tilde{\delta} := \delta_2^{-1} \circ \delta_1: \mathbf{B}_1 \rightarrow \mathbf{B}_2$ , we have  $\beta_1 \cong \beta_2 \circ H^0(\tilde{\delta})$ .

In view of Example 2.7, it is straightforward to deduce the following special instance of Theorem 1.2.

**Corollary 4.6.** *Let  $X$  be a quasi-projective scheme containing a projective subscheme  $Z$  such that  $T_0(\mathcal{O}_Z) = 0$  and either  $X$  is smooth or  $X = Z$ . Then  $\mathbf{Perf}_Z(X)$  has a strongly unique enhancement.*

If  $X = Z$ , then this is nothing but one of the main results in [24] (see Theorem 9.9 there).

**4.3. Fourier–Mukai kernels.** Let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  such that  $\mathcal{O}_{iZ_1} \in \mathbf{Perf}(X_1)$ , for all  $i > 0$ . Assume that  $X_2$  is a scheme containing a subscheme  $Z_2$  proper over  $\mathbb{k}$ . Let  $F: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  be an exact functor satisfying  $(*)$ .

To construct the Fourier–Mukai kernel realizing  $F$  as a Fourier–Mukai functor we will make use of some ideas from Sections 4, 6 and 9 of [24].

**4.3.1. The dg-lift.** For  $i = 1, 2$ , let  $\mathbf{Perf}_{Z_i}^{\mathrm{dg}}(X_i)$  be the full dg-subcategory of the enhancement  $\mathbf{D}_{Z_i}^{\mathrm{dg}}(X_i)$  of the triangulated category  $\mathbf{D}_{Z_i}(\mathbf{Qcoh}(X_i))$  consisting of perfect objects (see Examples 4.2 and 4.3). Keep moreover the same notation as in Section 4.2.

**Remark 4.7.** The exact functors  $\iota: \mathbf{D}_Z(\mathbf{Qcoh}(X)) \rightarrow \mathbf{D}(\mathbf{Qcoh}(X))$  and  $\iota^!: \mathbf{D}(\mathbf{Qcoh}(X)) \rightarrow \mathbf{D}_Z(\mathbf{Qcoh}(X))$ , defined in Section 2.1, have natural dg-lifts (denoted with the same symbols)  $\iota: \mathbf{D}_Z^{\mathrm{dg}}(X) \rightarrow \mathbf{D}^{\mathrm{dg}}(X)$  and  $\iota^!: \mathbf{D}^{\mathrm{dg}}(X) \rightarrow \mathbf{D}_Z^{\mathrm{dg}}(X)$ . For this use the enhancement via h-injective complexes as in the proof of [24, Thm. 7.9] and Lemma 4.4.

Set  $F' := F \circ \varphi^{-1}|_{(\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c}$ , where  $\varphi$  is the equivalence in Lemma 4.4. Since the objects in the essential image of  $\pi \circ h$  form a set of compact objects in  $\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L}$ , define the full dg-subcategory  $\mathbf{B} := \{F'(\pi(h^P)) : P \in \mathbf{A}\}$  of  $\mathbf{D}_{Z_2}^{\mathrm{dg}}(X_2)$ . Obviously, the objects of  $\mathbf{B}$  are compact in  $\mathbf{D}_{Z_2}^{\mathrm{dg}}(X_2)$ .

Let  $\mathbf{C}$  be a dg-subcategory of  $\mathbf{Perf}_{Z_2}^{\mathrm{dg}}(X_2)$  such that  $H^0(\mathbf{C})$  is classically generated by the objects in  $\mathbf{B}$ . By definition the functor  $F': (\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  factors through  $H^0(\mathbf{C})$  giving rise to the functor

$$\mathbf{A} \xrightarrow{\pi \circ h} (\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c \xrightarrow{F'} H^0(\mathbf{C}) \hookrightarrow \mathbf{Perf}_{Z_2}(X_2)$$

which, in turn, factors through  $H^0(\mathbf{B})$ . Hence we can consider the dg-functor

$$(4.4) \quad \rho_1: \mathbf{A} \xrightarrow{\pi \circ h} (\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c \rightarrow H^0(\mathbf{B}) \hookrightarrow \mathbf{D}^{\mathrm{dg}}(\mathbf{B}).$$

Consider the dg-category  $\tau_{\leq 0}\mathbf{B}$  with the same objects as  $\mathbf{B}$  but such that  $\mathrm{Hom}_{\tau_{\leq 0}\mathbf{B}}(\mathcal{E}, \mathcal{F}) = \tau_{\leq 0}\mathrm{Hom}_{\mathbf{B}}(\mathcal{E}, \mathcal{F})$  (here  $\tau_{\leq 0}$  is the gentle truncation). Let  $p: \tau_{\leq 0}\mathbf{B} \rightarrow H^0(\mathbf{B})$  be the natural functor. Due to Lemma 4.4 and assumption  $(*)$  (see in particular item (1)),  $p$  is a quasi-equivalence. Thus from (4.4) we get the quasi-functor

$$\rho_2: \mathrm{SF}(\mathbf{A}) \xrightarrow{\rho_1^*} \mathrm{SF}(H^0(\mathbf{B})) \xrightarrow{p_*} \mathrm{SF}(\tau_{\leq 0}\mathbf{B}) \rightarrow \mathrm{SF}(\mathbf{B}).$$

By [24, Lemma 6.2], the quasi-functor  $\rho_2$  factors through the dg-quotient  $\mathrm{SF}(\mathbf{A})/(\mathrm{SF}(\mathbf{A}) \cap \mathbf{L}')$ , where  $\mathbf{L}'$  is the lift of the localizing category  $\mathbf{L}$  to  $\mathrm{SF}(\mathbf{A})$ . Hence we get a quasi-functor

$$\rho_3: \mathrm{SF}(\mathbf{A})/(\mathrm{SF}(\mathbf{A}) \cap \mathbf{L}') \rightarrow \mathrm{SF}(\mathbf{B}).$$

(Notice that in the proof of [24, Lemma 6.2], the assumption that  $F$  is fully faithful is not needed.)

Let  $\mathbf{Perf}^{\mathrm{dg}}(\mathbf{A})$  and  $\mathbf{Perf}^{\mathrm{dg}}(\mathbf{B})$  be, respectively, the full dg-subcategories of  $\mathrm{SF}(\mathbf{A})/(\mathrm{SF}(\mathbf{A}) \cap \mathbf{L}')$  and  $\mathrm{SF}(\mathbf{B})$  consisting of the compact objects  $(\mathbf{D}^{\mathrm{dg}}(\mathbf{A})/\mathbf{L})^c$  and  $\mathbf{D}^{\mathrm{dg}}(\mathbf{B})^c$ . By its definition, the

quasi-functor  $\rho_3$  restricts to a quasi-functor  $\rho_3: \text{Perf}^{\text{dg}}(\mathbf{A}) \rightarrow \text{Perf}^{\text{dg}}(\mathbf{B})$ . The same argument as in the proof of [24, Prop. 1.16] applies and gives a quasi-equivalence  $\psi: \mathbf{C} \rightarrow \text{Perf}^{\text{dg}}(\mathbf{B})$ . Set

$$(4.5) \quad \mathbf{F}^{\text{dg}}: \text{Perf}^{\text{dg}}(\mathbf{A}) \xrightarrow{\rho_3} \text{Perf}^{\text{dg}}(\mathbf{B}) \xrightarrow{\psi^{-1}} \mathbf{C} \hookrightarrow \text{Perf}_{Z_2}^{\text{dg}}(X_2).$$

Due to the uniqueness of the enhancement of  $D_{Z_1}(\mathbf{Qcoh}(X_1))$ , the quasi-functor (4.5), yields a quasi-functor

$$\mathbf{F}_1^{\text{dg}}: \text{Perf}_{Z_1}^{\text{dg}}(X_1) \longrightarrow \text{Perf}_{Z_2}^{\text{dg}}(X_2)$$

4.3.2. *The isomorphism.* Consider the exact functor

$$H^0(\mathbf{F}^{\text{dg}}): (D^{\text{dg}}(\mathbf{A})/\mathbf{L})^c \longrightarrow \mathbf{Perf}_{Z_2}(X_2)$$

and the composition  $\mathbf{F}_1 := H^0(\mathbf{F}^{\text{dg}}) \circ \varphi|_{\mathbf{Perf}_{Z_1}(X_1)}$ . Notice that  $H^0(\mathbf{F}_1^{\text{dg}}) \cong \mathbf{F}_1$ .

As a consequence of [24, Lemma 6.1] (see also [24, Prop. 3.4]) and of the definition in (4.5), we get an isomorphism

$$\theta_1: \mathbf{F} \circ \varphi^{-1} \circ \pi \circ h \xrightarrow{\sim} H^0(\mathbf{F}^{\text{dg}}) \circ \pi \circ h,$$

as functors from  $\mathbf{A}$  to  $\mathbf{Perf}_{Z_2}(X_2)$ .

As, by Lemma 4.4,  $\varphi^{-1}(\pi(h^P)) = P$  for any object  $P$  in the weakly ample set  $\mathbf{A}$ , the isomorphism  $\theta_1$  gives

$$\theta_2: \mathbf{F}|_{\mathbf{A}} \xrightarrow{\sim} \mathbf{F}_1|_{\mathbf{A}}.$$

Applying Corollary 3.8 we get an isomorphism of exact functors

$$(4.6) \quad \theta: \mathbf{F} \xrightarrow{\sim} \mathbf{F}_1.$$

4.3.3. *The kernel.* By [24, Prop. 1.17], we get a quasi-equivalence  $\varphi_i: \text{SF}(\text{Perf}_{Z_i}^{\text{dg}}(X_i)) \xrightarrow{\sim} D_{Z_i}^{\text{dg}}(X_i)$ . Hence, composing the extension of  $\mathbf{F}_1^{\text{dg}}$  to semi-free modules with  $\varphi_i$ , we get a quasi-functor

$$\mathbf{F}_2^{\text{dg}}: D_{Z_1}^{\text{dg}}(X_1) \xrightarrow{\varphi_1^{-1}} \text{SF}(\text{Perf}_{Z_1}^{\text{dg}}(X_1)) \rightarrow \text{SF}(\text{Perf}_{Z_2}^{\text{dg}}(X_2)) \xrightarrow{\varphi_2} D_{Z_2}^{\text{dg}}(X_2)$$

which, by definition, commutes with direct sums (essentially because it has a right adjoint, according to [24, Sect. 1]). Observe that  $H^0(\mathbf{F}_1^{\text{dg}}) \cong H^0(\mathbf{F}_2^{\text{dg}})|_{\mathbf{Perf}_{Z_1}(X_1)}$ .

The easier case  $X_i = Z_i$ , for  $i = 1, 2$ , generalizing one of the main results in [24], can be treated already.

**Corollary 4.8.** *Let  $X_1$  be a projective variety and let  $X_2$  be a scheme. For any exact functor  $\mathbf{F}: \mathbf{Perf}(X_1) \rightarrow \mathbf{Perf}(X_2)$  satisfying (\*), there exists  $\mathcal{E} \in D^b(X_1 \times X_2)$  and an isomorphism  $\mathbf{F} \cong \Phi_{\mathcal{E}}$ .*

*Proof.* By [27, Thm. 8.9] there is  $\mathcal{E} \in D(\mathbf{Qcoh}(X_1 \times X_2))$  such that  $\Phi_{\mathcal{E}} \cong H^0(\mathbf{F}_2^{\text{dg}})$ . As  $\mathbf{F}_1 \cong H^0(\mathbf{F}_1^{\text{dg}}) \cong H^0(\mathbf{F}_2^{\text{dg}})|_{\mathbf{Perf}(X_1)}$ , the isomorphism (4.6) gives  $\mathbf{F} \cong \Phi_{\mathcal{E}}$ . The fact that  $\mathcal{E}$  is bounded coherent is obtained by the same argument as in the proof of [24, Cor. 9.13], part (4). We do not explain this here as this is a special instance of Lemma 5.2.  $\square$

Back to the general setting, let  $F_3^{\text{dg}}$  be the quasi-functor making the following diagram commutative

$$(4.7) \quad \begin{array}{ccc} \mathrm{D}^{\mathrm{dg}}(X_1) & \xrightarrow{F_3^{\mathrm{dg}}} & \mathrm{D}^{\mathrm{dg}}(X_2) \\ \iota^! \downarrow & & \uparrow \iota \\ \mathrm{D}_{Z_1}^{\mathrm{dg}}(X_1) & \xrightarrow{F_2^{\mathrm{dg}}} & \mathrm{D}_{Z_2}^{\mathrm{dg}}(X_2). \end{array}$$

As  $F_2^{\text{dg}}$  commutes with direct sums,  $F_3^{\text{dg}}$  does. Notice that, if  $Z_1 = X_1$ , then  $\iota^! = \text{id}$ .

By [27, Thm. 8.9], there exists  $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$  and an isomorphism of exact functors

$$\Phi_{\mathcal{E}} \cong H^0(F_3^{\text{dg}}): \mathrm{D}(\mathbf{Qcoh}(X_1)) \xrightarrow{\iota^!} \mathrm{D}_{Z_1}(\mathbf{Qcoh}(X_1)) \xrightarrow{H^0(F_2^{\text{dg}})} \mathrm{D}_{Z_2}(\mathbf{Qcoh}(X_2)) \xrightarrow{\iota} \mathrm{D}(\mathbf{Qcoh}(X_2))$$

such that  $\iota \circ H^0(F_2^{\text{dg}})(\mathcal{A}) \cong \Phi_{\mathcal{E}} \circ \iota(\mathcal{A})$ , for any  $\mathcal{A} \in \mathrm{D}_{Z_1}^b(X_1)$ .

**Lemma 4.9.** *Under the above assumptions, there exists an isomorphism of exact functors*

$$H^0(F_2^{\text{dg}})|_{\mathbf{Perf}_{Z_1}(X_1)} \cong \Phi_{\tilde{\mathcal{E}}}^s|_{\mathbf{Perf}_{Z_1}(X_1)},$$

where  $\tilde{\mathcal{E}} = (\iota \times \iota)^! \mathcal{E} \in \mathrm{D}_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$ .

*Proof.* For all  $\mathcal{A} \in \mathbf{Perf}_{Z_1}(X_1)$  and  $\mathcal{B} \in \mathbf{Perf}_{Z_2}(X_2)$  we have the following isomorphisms

$$\begin{aligned} \mathrm{Hom}(\mathcal{B}, \Phi_{\tilde{\mathcal{E}}}^s(\mathcal{A})) &= \mathrm{Hom}(\mathcal{B}, \iota^!(p_2)_*((\iota \times \iota)(\iota \times \iota)^! \mathcal{E} \otimes p_1^*(\mathcal{A}))) \\ &\cong \mathrm{Hom}((\iota \mathcal{A})^\vee \boxtimes \iota \mathcal{B}, (\iota \times \iota)(\iota \times \iota)^! \mathcal{E}), \end{aligned}$$

where we are using the adjunction between  $\iota$  and  $\iota^!$ . Observe that, if  $Z_1 = X_1$ , then the functor  $\iota$  for  $X_1$  is the identity.

As  $(\iota \mathcal{A})^\vee \boxtimes \iota \mathcal{B}$  is supported on  $Z_1 \times Z_2$  and  $\iota \times \iota$  is fully faithful, the latter vector space is isomorphic to

$$\mathrm{Hom}((\iota \mathcal{A})^\vee \boxtimes \iota \mathcal{B}, \mathcal{E}) \cong \mathrm{Hom}(\mathcal{B}, \iota^! \Phi_{\mathcal{E}}(\iota \mathcal{A})).$$

Hence  $\Phi_{\tilde{\mathcal{E}}}^s \cong \iota^! \circ \Phi_{\mathcal{E}} \circ \iota \cong H^0(F_2^{\text{dg}})$  (use the commutativity of (4.7) and the fact that  $\iota^! \circ \iota = \text{id}$ ).  $\square$

Putting this together with the isomorphism (4.6) in Section 4.3.2, we have proved the following.

**Proposition 4.10.** *Let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  such that  $\mathcal{O}_{iZ_1} \in \mathbf{Perf}(X_1)$ , for all  $i > 0$ . Assume that  $X_2$  is a scheme containing a subscheme  $Z_2$  proper over  $\mathbb{k}$ .*

*Then, for any exact functor  $F: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  satisfying (\*) there exist  $\mathcal{E} \in \mathrm{D}_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$  and an isomorphism of exact functors  $F \cong \Phi_{\mathcal{E}}^s$ .*

By Example 2.7, the following consequence immediately holds true.

**Corollary 4.11.** *Let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  such that either  $X_1$  is smooth or  $X_1 = Z_1$ . Assume that  $X_2$  is a scheme containing a subscheme  $Z_2$  proper over  $\mathbb{k}$ .*

*Then, for any exact functor  $F: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  satisfying (\*) there exist  $\mathcal{E} \in \mathrm{D}_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$  and an isomorphism of exact functors  $F \cong \Phi_{\mathcal{E}}^s$ .*

Using part (i) of Example 3.12 for  $X_1 = Z_1$ , this corollary and Corollary 4.8 may be seen as generalizations of [24, Cor. 9.13] (see, in particular, parts (2) and (3) there).

The following example will be important in the rest of the paper.

**Example 4.12. (The Fourier–Mukai kernel of the identity.)** We want to show that a Fourier–Mukai kernel of the identity functor  $\text{id}: D_{Z_1}^b(X_1) \rightarrow D_{Z_1}^b(X_1)$  is

$$(\iota \times \iota)^! \mathcal{I} \in D_{Z_1 \times Z_1}^b(\mathbf{Qcoh}(X_1 \times X_1)),$$

where, denoting by  $\Delta: X_1 \rightarrow X_1 \times X_1$  the diagonal embedding,

$$\mathcal{I} := \Delta_* \circ \iota \circ \iota^!(\mathcal{O}_{X_1}).$$

Indeed, it is enough to prove that  $\iota^! \Phi_{\mathcal{I}}(\iota \mathcal{B}) \cong \mathcal{B}$ , for any  $\mathcal{B} \in D_{Z_1}^b(X_1)$ . But, as in the proof of Lemma 4.9, this is a consequence of the following isomorphisms:

$$\begin{aligned} \text{Hom}(\mathcal{A}, \iota^! \Phi_{\mathcal{I}}(\iota \mathcal{B})) &\cong \text{Hom}(\iota \mathcal{A}, (p_2)_*(\Delta_* \circ \iota \circ \iota^!(\mathcal{O}_{X_1}) \otimes p_1^*(\iota \mathcal{B}))) \\ &\cong \text{Hom}((\iota \mathcal{B})^\vee \boxtimes \iota \mathcal{A}, \Delta_* \circ \iota \circ \iota^!(\mathcal{O}_{X_1})) \\ &\cong \text{Hom}((\iota \mathcal{B})^\vee \otimes \iota \mathcal{A}, \mathcal{O}_{X_1}) \\ &\cong \text{Hom}(\mathcal{A}, \iota^! \mathcal{B}) \cong \text{Hom}(\mathcal{A}, \mathcal{B}), \end{aligned}$$

for any  $\mathcal{A} \in D_{Z_1}^b(X_1)$ . Here  $p_i: X_1 \times X_1 \rightarrow X_1$  is the natural projection. Again, for the first and the fourth isomorphism we used the adjunction between  $\iota$  and  $\iota^!$ . The same adjunction together with the one between  $\Delta^*$  and  $\Delta_*$  and the fact that  $\iota$  is fully faithful and  $(\iota \mathcal{B})^\vee \otimes \iota \mathcal{A}$  has support in  $Z_1$  explains the third isomorphism.

**4.4. The category generated by a spherical object.** Let  $\mathbf{T}$  be an algebraic triangulated category over an algebraically closed field  $\mathbb{k}$  (see [21] for the definition of algebraic triangulated category) and let  $S \in \mathbf{T}$  be a  $d$ -spherical object. As we mentioned in the introduction, this means that  $\text{Hom}_{\mathbf{T}}(S, S[i])$  is trivial if  $i \neq 0, d$  ( $d$  is a positive integer) while it is isomorphic to  $\mathbb{k}$  otherwise. Denote by  $\mathbf{T}_S$  the triangulated subcategory of  $\mathbf{T}$  generated by  $S$ .

**Remark 4.13.** In the case  $d = 1$ , for any smooth curve  $C$  and any closed point  $p \in C$ , there is an exact equivalence  $\mathbf{T}_S \cong D_p^b(C)$ . Moreover,  $\mathbf{T}_S$  is a 1-Calabi–Yau category. Indeed, the equivalence in the statement is a straightforward consequence of [22, Thm. 2.1]. Obviously the Serre functor of  $D_p^b(C)$  is the shift by one and so  $\mathbf{T}_S$  is a 1-Calabi–Yau category.

For those categories we can prove variants of Theorems 1.1 and 1.2. Notice that in this case it is not true that the maximal 0-dimensional torsion subsheaf of  $\mathcal{O}_Z$  is trivial. In particular, as remarked in [24, Prop. 9.2], the construction in Section 2.2 does not provide a weakly ample set. Thus we need a particular treatment that, unfortunately, works only for points embedded in curves.

**Proposition 4.14.** *Let  $p$  be a closed point in a smooth projective curve  $C$*

(i) *Let  $X$  be a separated scheme of finite type over  $\mathbb{k}$  with a subscheme  $Z$  which is proper over  $\mathbb{k}$  and let*

$$F: D_p^b(C) \longrightarrow \mathbf{Perf}_Z(X)$$

be an exact functor such that

$$(4.8) \quad \mathrm{Hom}_{\mathbf{Perf}_Z(X)}(\mathbf{F}(\mathcal{A}), \mathbf{F}(\mathcal{B})[k]) = 0,$$

for all  $\mathcal{A}, \mathcal{B} \in \mathbf{Coh}_p(C)$  and all  $k < 0$ . Then there exist  $\mathcal{E} \in D_{\{p\} \times Z}^b(\mathbf{Qcoh}(C \times X))$  and an isomorphism of exact functors  $\mathbf{F} \cong \Phi_{\mathcal{E}}^s$ .

(ii) The triangulated category  $D_p^b(C)$  has a strongly unique enhancement.

*Proof.* The category  $\mathbf{C}$ , whose objects are  $\{\mathcal{O}_{n_p} : n > 0\}$ , satisfies properties (1) and (2) in Definition 2.5. In particular, looking carefully at the construction in Section 4.3, this together with (4.8) is enough to provide an  $\mathcal{E} \in D_{\{p\} \times Z_2}^b(\mathbf{Qcoh}(C \times X))$  and an isomorphism

$$\theta: \mathbf{F}|_{\mathbf{C}} \xrightarrow{\sim} \Phi_{\mathcal{E}}^s|_{\mathbf{C}}.$$

To be precise, the fact that  $\mathcal{E}$  is a bounded complex is a consequence of Lemma 5.3 below.

Set  $\mathbf{D}_0$  to be the (strictly) full subcategory of  $D_p^b(X)$  whose objects are isomorphic to shifts of objects of  $\mathbf{Coh}_p(X)$ . By Corollary 3.6, the isomorphism  $\theta$  extends (uniquely) to an isomorphism compatible with shifts

$$\theta_0: \mathbf{F}|_{\mathbf{D}_0} \xrightarrow{\sim} \Phi_{\mathcal{E}}^s|_{\mathbf{D}_0}.$$

Being  $C$  a smooth curve, any object  $\mathcal{F} \in D_p^b(C)$  can be written (in an essentially unique way) as a finite direct sum of objects of  $\mathbf{D}_0$ . Thus  $\theta_0$  extends (uniquely) to the desired isomorphism  $\mathbf{F} \xrightarrow{\sim} \Phi_{\mathcal{E}}^s$  proving (i).

As for (ii), observe that the first part of the proof of Theorem 1.2 in Section 4.2 carries over also in this case. The main difference is that the extension of the isomorphism (4.3) takes place due to Corollary 3.6 instead of Corollary 3.8, and then reasoning as in the last part of the proof of (i).  $\square$

## 5. UNIQUENESS OF FOURIER–MUKAI KERNELS

For functors satisfying (\*) and with  $X_1$  and  $X_2$  smooth, the uniqueness of Fourier–Mukai kernels is proved via a direct computation in Section 5.2. As a preliminary step, we study some basic properties of Fourier–Mukai functors in the supported setting.

**5.1. Basic properties.** Let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  and assume that  $X_2$  is a scheme containing a subscheme  $Z_2$  proper over  $\mathbb{k}$ . As explained in the following example, we cannot expect that the Fourier–Mukai kernel  $\mathcal{E}$  of a functor  $\Phi_{\mathcal{E}}: D_{Z_1}^b(X_1) \rightarrow D_{Z_2}^b(X_2)$  is coherent whenever  $Z_i \neq X_i$ .

**Example 5.1.** Assume that  $Z$  is a non-trivial smooth projective subvariety of a smooth quasi-projective variety  $X$ . Suppose that there exists  $\mathcal{E} \in D_{Z \times Z}^b(X \times X)$  such that

$$\Phi_{\mathcal{E}}^s \cong \mathrm{id}: D_Z^b(X) \rightarrow D_Z^b(X).$$

By [26, Lemma 7.41], there exist  $n > 0$  and  $\mathcal{E}_n \in D^b(nZ \times nZ)$  such that  $(\iota \times \iota)\mathcal{E} \cong (i_n \times i_n)_*\mathcal{E}_n$ , where  $i_n: nZ \rightarrow X$  is the embedding. For any  $\mathcal{F}_n \in D^b(nZ)$ , we have

$$(5.1) \quad (i_n)_*\mathcal{F}_n \cong \Phi_{\mathcal{E}}((i_n)_*\mathcal{F}_n) \cong (i_n)_*\Phi_{\mathcal{E}_n}((i_n)^*(i_n)_*\mathcal{F}_n).$$

Take now  $X = \mathbb{P}^k$ ,  $Z = \mathbb{P}^{k-1}$  and  $\mathcal{F}_n := \mathcal{O}_{nZ}(m)$ , for  $m \in \mathbb{Z}$ . An easy calculation shows that  $(i_n)^*(i_n)_*\mathcal{F}_n \cong \mathcal{O}_{nZ}(m) \oplus \mathcal{O}_{nZ}(m-n)[1]$ . Hence to have (5.1) verified, we should have either

$\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m)) = 0$  or  $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m-n)) = 0$ . But the following isomorphisms should hold at the same time

$$\begin{aligned}\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m)) \oplus \Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m-n))[1] &\cong \mathcal{O}_{nZ}(m), \\ \Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m+n)) \oplus \Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m))[1] &\cong \mathcal{O}_{nZ}(m+n).\end{aligned}$$

If  $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m-n)) = 0$ , then from the second one we would have that  $\mathcal{O}_{nZ}(m)[1]$  is a direct summand of  $\mathcal{O}_{nZ}(m+n)$  which is absurd. Thus  $\Phi_{\mathcal{E}_n}(\mathcal{O}_{nZ}(m)) = 0$ . As this holds for all  $m \in \mathbb{Z}$ , we get a contradiction.

Nevertheless, by construction, a functor  $F: D_{Z_1}^b(X_1) \rightarrow D_{Z_2}^b(X_2)$  satisfying  $(*)$  is a Fourier–Mukai functor with kernel in  $D_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$ . We now want to prove that such a complex has bounded cohomologies.

We start by recalling that a functor

$$F: D_{Z_1}^b(\mathbf{Qcoh}(X_1)) \rightarrow D_{Z_2}^b(\mathbf{Qcoh}(X_2))$$

is *bounded* if there is an interval  $[a, b] \subset \mathbb{R}$  such that, for any  $\mathcal{A} \in \mathbf{Qcoh}_{Z_1}(X_1)$ , we have that if  $H^i(F(\mathcal{A})) \neq 0$ , then  $i \in [a, b]$ . We begin with the following rather standard lemma.

**Lemma 5.2.** *Assume that the base field  $\mathbb{k}$  is perfect. Then every exact functor  $F: D_{Z_1}^b(X_1) \rightarrow D_{Z_2}^b(\mathbf{Qcoh}(X_2))$  or  $G: D_{Z_1}^b(\mathbf{Qcoh}(X_1)) \rightarrow D_{Z_2}^b(\mathbf{Qcoh}(X_2))$  is bounded.*

*Proof.* To deal with the first part, observe that, by Proposition 2.4, the category  $\mathbf{Coh}_{Z_1}(X_1)$  is generated, as an abelian category, by the image of the natural fully faithful functor  $i_*: \mathbf{Coh}(Z_1) \hookrightarrow \mathbf{Coh}_{Z_1}(X_1)$ . This means that it is enough to show the boundedness of the functor

$$F'' := F \circ i_*: D^b(Z_1) \rightarrow D_{Z_2}^b(\mathbf{Qcoh}(X_2)).$$

Now this is a straightforward consequence of [26, Thm. 7.39] (here we need that  $\mathbb{k}$  is perfect). The second part, concerning exact functors between the bounded derived categories of quasi-coherent sheaves, is proved using the same argument.  $\square$

We can also prove the following.

**Lemma 5.3.** *Let  $X_1$  be a quasi-projective scheme containing a projective subscheme  $Z_1$  and let  $X_2$  be a scheme containing a subscheme  $Z_2$  proper over  $\mathbb{k}$ .*

*If  $\Phi_{\mathcal{E}}^s: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  is a Fourier–Mukai functor, then  $\mathcal{E} \in D_{Z_1 \times Z_2}^b(\mathbf{Qcoh}(X_1 \times X_2))$ . Moreover, if  $X_1 = Z_1$ , then  $\mathcal{E} \in D_{X_1 \times Z_2}^b(X_1 \times X_2)$ .*

*Proof.* By [26, Thm. 6.8], for  $i = 1, 2$ , the category  $D_{Z_i}(\mathbf{Qcoh}(X_i))$  has a compact generator  $G_i \in \mathbf{Perf}_{Z_i}(X_i)$  (see [5] for the non-supported case). Moreover, by the explicit description of the compact generator in the proof of [26, Thm. 6.8], one sees that  $G_1 \boxtimes G_2$  is a compact generator of  $D_{Z_1 \times Z_2}(\mathbf{Qcoh}(X_1 \times X_2))$ .

By [26, Prop. 6.9], the kernel  $\mathcal{E}$  has bounded cohomology if and only if there exists an interval  $[a, b] \subset \mathbb{R}$  such that  $\mathrm{Hom}(G_1 \boxtimes G_2, \mathcal{E}[k]) = 0$ , for any  $k \notin [a, b]$ . But now

$$\mathrm{Hom}(G_1 \boxtimes G_2, \mathcal{E}[k]) \cong \mathrm{Hom}(\iota G_1 \boxtimes \iota G_2, (\iota \times \iota)\mathcal{E}[k]) \cong \mathrm{Hom}(G_2, \Phi_{\mathcal{E}}^s(G_1^\vee)[k])$$

which is non-trivial only for finitely many  $k \in \mathbb{Z}$ .

Suppose that  $Z_1 = X_1$ . Then  $\mathcal{E} \in D_{X_1 \times Z_2}^b(X_1 \times X_2)$  if and only if  $\mathcal{E}_1 := (\text{id} \times \iota)\mathcal{E} \in D^b(X_1 \times X_2)$ . Since an easy calculation shows that  $\iota \circ \Phi_{\mathcal{E}}^s \cong \Phi_{\mathcal{E}_1}$ , the functor  $\Phi_{\mathcal{E}_1}$  sends perfect complexes to perfect complexes. Hence we can assume, without loss of generality, that  $Z_i = X_i$ , for  $i = 1, 2$ . Then it follows from [24, Cor. 9.13 (4)] (see also [10, Lemma 3.7]) that  $\mathcal{E}_1 \in D^b(X_1 \times X_2)$ .  $\square$

**5.2. The uniqueness of the Fourier–Mukai kernels.** Assume that the base field  $\mathbb{k}$  is perfect and that  $X_1$  and  $X_2$  are smooth quasi-projective schemes containing projective subschemes  $Z_1$  and  $Z_2$ . Take a functor  $F: D_{Z_1}^b(X_1) \rightarrow D_{Z_2}^b(X_2)$  satisfying  $(*)$  and assume that there are  $\mathcal{E}_1, \mathcal{E}_2 \in D_{Z_1 \times Z_2}^b(\mathbf{Qcoh}(X_1 \times X_2))$  such that

$$F \cong \Phi_{\mathcal{E}_1}^s \cong \Phi_{\mathcal{E}_2}^s.$$

Obviously  $\mathcal{E}_i \cong (\iota \times \iota)^!(\iota \times \iota)\mathcal{E}_i$ , for  $i = 1, 2$ .

Set  $\mathcal{F}_i := (\iota \times \iota)\mathcal{E}_i \in D^b(\mathbf{Qcoh}(X_1 \times X_2))$ . We get a Fourier–Mukai functor

$$\Phi_i := \Phi_{\mathcal{O}_{\Delta} \boxtimes \mathcal{F}_i}: D(\mathbf{Qcoh}(X_1 \times X_1)) \longrightarrow D(\mathbf{Qcoh}(X_1 \times X_2)),$$

for  $i = 1, 2$ . Let  $\mathcal{I}$  be the complex in Example 4.12 such that  $(\iota \times \iota)^!\mathcal{I}$  is the Fourier–Mukai kernel of the identity  $\text{id}: D_{Z_1}^b(X_1) \rightarrow D_{Z_1}^b(X_1)$ . We first prove the following result.

**Lemma 5.4.** *For  $i = 1, 2$  and  $P_1, P_2 \in \mathbf{Amp}(Z_1, X_1, H_1)$ , we have  $\Phi_i(\mathcal{I}) \cong \mathcal{F}_i$  and  $\Phi_i(\iota P_1 \boxtimes \iota P_2) \cong \iota P_1 \boxtimes \Phi_{\mathcal{F}_i}(\iota P_2)$ .*

*Proof.* An easy calculation using the projection formula and base change yields

$$\Phi_i(\mathcal{I}) \cong (p_1)^* \iota^! \mathcal{O}_{X_1} \otimes \mathcal{F}_i,$$

where  $p_1: X_1 \times X_2 \rightarrow X_1$  is the projection. Then we have the following sequence of isomorphisms

$$\begin{aligned} \Phi_i(\mathcal{I}) &\cong (p_1)^* \underset{\rightarrow n}{\text{colim}} \mathbf{R}\mathcal{H}om(\mathcal{O}_{nZ_1}, \mathcal{O}_{X_1}) \otimes \mathcal{F}_i \\ &\cong \underset{\rightarrow n}{\text{colim}} \mathbf{R}\mathcal{H}om(\mathcal{O}_{nZ_1 \times X_2}, \mathcal{O}_{X_1 \times X_2}) \otimes \mathcal{F}_i \\ &\cong \underset{\rightarrow n}{\text{colim}} \mathbf{R}\mathcal{H}om(\mathcal{O}_{nZ_1 \times X_2}, \mathcal{F}_i) \\ &\cong (\iota \times \text{id})(\iota \times \text{id})^! \mathcal{F}_i \\ &\cong \mathcal{F}_i. \end{aligned}$$

The second assertion in the statement is clear.  $\square$

As  $\mathcal{I}$  is a bounded complex of quasi-coherent sheaves, there exists a bounded above complex  $\mathcal{L}^\bullet \in D(\mathbf{Qcoh}(X_1 \times X_1))$  such that  $\mathcal{L}^\bullet \cong \mathcal{I}$  and, for any  $j \in \mathbb{Z}$ , the sheaf  $\mathcal{L}^j$  in  $j$ -th position in  $\mathcal{L}$  is of the form  $P_j \boxtimes M_j$ , where  $P_j$  and  $M_j$  are (possibly infinite) direct sums of sheaves in  $\mathbf{Amp}(Z_1, X_1, H_1)$  (use Proposition 2.8).

Being  $\mathcal{I}$  bounded, for  $m > 0$  sufficiently large, the stupid truncation  $\mathcal{M}^\bullet := \sigma_{\geq -m} \mathcal{L}^\bullet$  of  $\mathcal{L}^\bullet$  in position  $-m$  is such that  $\mathcal{M}^\bullet \cong \mathcal{I} \oplus \mathcal{K}[m]$ , for some  $\mathcal{K} \in \mathbf{Qcoh}(X_1 \times X_1)$ . Applying term by term the functor  $\Phi_i$  to  $\mathcal{M}^\bullet$  we get a complex of complexes

$$\Phi_i(\mathcal{L}^{-m}) \longrightarrow \cdots \longrightarrow \Phi_i(\mathcal{L}^0).$$

Due to Lemma 5.4, the choice of  $m$  sufficiently large, the assumption  $(*)$  and Lemma 3.1, this complex has a unique (up to isomorphism) right convolution  $\mathcal{A}_i := \mathcal{F}_i \oplus \mathcal{K}_i[m]$ , with  $\mathcal{K}_i = \Phi_i(\mathcal{K}) \in D_{Z_1 \times Z_2}^b(\mathbf{Qcoh}(X_1 \times X_2))$ .

Applying Lemma 3.2 under the hypothesis  $(*)$ , we get  $\mathcal{A}_1 \cong \mathcal{A}_2$ . By Lemma 5.2 (here we use that  $\mathbb{k}$  is perfect), the functor  $\Phi_i$  is bounded and so, for  $m$  large enough,  $\mathrm{Hom}(\mathcal{F}_1, \mathcal{K}_2[m]) \cong \mathrm{Hom}(\mathcal{K}_1[m], \mathcal{F}_2) \cong 0$ . Hence,

$$(\iota \times \iota)\mathcal{E}_1 = \mathcal{F}_1 \cong \mathcal{F}_2 = (\iota \times \iota)\mathcal{E}_2$$

which is equivalent to  $\mathcal{E}_1 \cong \mathcal{E}_2$ .

Hence we proved the following result which is precisely the uniqueness statement in Theorem 1.1.

**Proposition 5.5.** *Let  $X_1, X_2, Z_1$  and  $Z_2$  be as above and let  $\Phi_{\mathcal{E}}^s: \mathbf{Perf}_{Z_1}(X_1) \rightarrow \mathbf{Perf}_{Z_2}(X_2)$  be a Fourier–Mukai functor satisfying  $(*)$ . Then  $\mathcal{E} \in \mathrm{D}_{Z_1 \times Z_2}^b(\mathbf{Qcoh}(X_1 \times X_2))$  is unique, up to isomorphism.*

**Remark 5.6.** Following a suggestion of D. Orlov, we can show that if  $X_1$  is a projective scheme such that  $T_0(\mathcal{O}_{X_1}) = 0$ ,  $X_2$  is a scheme and  $\Phi_{\mathcal{E}}: \mathbf{Perf}(X_1) \rightarrow \mathrm{D}^b(X_2)$  is an exact fully faithful functor, then  $\mathcal{E} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$  is uniquely determined (up to isomorphism).

Indeed, suppose that there exist  $\mathcal{F} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_2))$  and an isomorphism  $\Phi_{\mathcal{F}} \cong \Phi_{\mathcal{E}}$ . Consider the dg-lifts

$$\Phi_{\mathcal{E}}^{\mathrm{dg}}, \Phi_{\mathcal{F}}^{\mathrm{dg}}: \mathrm{Perf}^{\mathrm{dg}}(X_1) \longrightarrow \mathrm{D}^{\mathrm{dg}}(X_2)$$

of the functors above. Let  $\mathbf{C} \subseteq \mathrm{D}^{\mathrm{dg}}(\mathbf{Qcoh}(X_2))$  be the full dg-subcategory whose objects are the same as those in the essential image of  $\Phi_{\mathcal{E}}$ . If  $\Psi^{\mathrm{dg}}: \mathbf{C} \rightarrow \mathrm{Perf}^{\mathrm{dg}}(X_1)$  is a quasi-inverse of  $\Phi_{\mathcal{F}}^{\mathrm{dg}}$ , consider the composition

$$\mathbf{F}^{\mathrm{dg}} := \Psi^{\mathrm{dg}} \circ \Phi_{\mathcal{E}}^{\mathrm{dg}}: \mathrm{Perf}^{\mathrm{dg}}(X_1) \longrightarrow \mathrm{Perf}^{\mathrm{dg}}(X_1)$$

which has the property  $\Phi_{\mathcal{O}_{\Delta}} \cong \mathrm{id} \cong H^0(\mathbf{F}^{\mathrm{dg}})$ . As in the proof of [24, Cor. 9.13], the dg-quasi-functor  $\mathbf{F}^{\mathrm{dg}}$  extends to  $\mathbf{G}^{\mathrm{dg}}: \mathrm{D}^{\mathrm{dg}}(X_1) \rightarrow \mathrm{D}^{\mathrm{dg}}(X_1)$ . On the other hand, by [27, Thm. 8.9], there exists (a unique)  $\mathcal{G} \in \mathrm{D}(\mathbf{Qcoh}(X_1 \times X_1))$  such that  $\mathbf{G}^{\mathrm{dg}} \cong \Phi_{\mathcal{G}}^{\mathrm{dg}}$ . Hence  $\Phi_{\mathcal{O}_{\Delta}} \cong H^0(\mathbf{F}^{\mathrm{dg}}) \cong \Phi_{\mathcal{G}}$  and, using for example [10, Thm. 1.2], we get  $\mathcal{G} \cong \mathcal{O}_{\Delta}$ . Therefore  $\mathbf{G}^{\mathrm{dg}} \cong \mathrm{id}$  and so  $\Phi_{\mathcal{E}}^{\mathrm{dg}} \cong \Phi_{\mathcal{F}}^{\mathrm{dg}}$ . Applying again [27, Thm. 8.9], we deduce  $\mathcal{E} \cong \mathcal{F}$ .

Notice that the proof above does not work if the functor  $\Phi_{\mathcal{E}}$  satisfies  $(*)$  in the introduction but it is not fully faithful. Nevertheless we expect the result to be true in this case as well.

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