

# SOME REMARKS ABOUT THE FM-PARTNERS OF K3 SURFACES WITH PICARD NUMBERS 1 AND 2

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ABSTRACT. In this paper we prove some results about K3 surfaces with Picard number 1 and 2. In particular, we give a new simple proof of a theorem due to Oguiso which shows that, given an integer  $N$ , there is a K3 surface with Picard number 2 and at least  $N$  non-isomorphic FM-partners. We describe also the Mukai vectors of the moduli spaces associated to the Fourier-Mukai partners of K3 surfaces with Picard number 1.

## 1. INTRODUCTION

In some recent papers Hosono, Lian, Oguiso and Yau (see [5] and [13]) gave a formula that counts the number of non-isomorphic Fourier-Mukai partners of a K3 surface. In this paper we are interested in the case of K3 surfaces with Picard number 1 and 2.

In the second paragraph, we recall the formula for the number of the isomorphism classes of Fourier-Mukai partners of a given K3 surface (given in [4]), which allows to count the isomorphism classes of Fourier-Mukai partners of a K3 surface with Picard number 1 (this is also given in [13]). As a first result, we will describe the Mukai vectors of the moduli spaces associated to the Fourier-Mukai partners of such K3 surfaces<sup>1</sup>. This gives some information about the geometry of the Fourier-Mukai partners of the given K3 surface.

In the third paragraph we prove that, given  $N$  and  $d$  positive integers, there is an elliptic K3 surface with a polarization of degree  $d$  and with at least  $N$  non-isomorphic elliptic Fourier-Mukai partners (Theorem 3.3). The most interesting consequence of this result is a new simple proof of Theorem 1.7 in [13] (Corollary 3.4 and Remark 3.5).

We start with recalling some essential facts about lattices and K3 surfaces.

**1.1. Lattices and discriminant groups.** A *lattice*  $L := (L, b)$  is a free abelian group of finite rank with a non-degenerate symmetric bilinear form  $b : L \times L \rightarrow \mathbb{Z}$ . Two lattices  $(L_1, b_1)$  and  $(L_2, b_2)$  are *isometric* if there is an isomorphism of abelian groups  $f : L_1 \rightarrow L_2$  such that  $b_1(x, y) = b_2(f(x), f(y))$ . We write  $O(L)$  for the group of all autoisometries of the lattice  $L$ . A lattice  $(L, b)$  is *even* if, for all  $x \in L$ ,  $x^2 := b(x, x) \in 2\mathbb{Z}$ , it is *odd* if there is  $x \in L$  such that  $b(x, x) \notin 2\mathbb{Z}$ . Given an integral basis for  $L$ , we can associate to the bilinear form a symmetric matrix  $S_L$  of dimension  $\text{rk}L$ , uniquely determined up to the action of  $\text{GL}(\text{rk}L, \mathbb{Z})$ . The integer  $\det L := \det S_L$  is called *discriminant* and it is an invariant of the lattice. A lattice is *unimodular* if  $\det L = \pm 1$ . Given  $(L, b)$  and  $k \in \mathbb{Z}$ ,  $L(k)$  is the lattice  $(L, kb)$ .

Given a sublattice  $V$  of  $L$  with  $V \hookrightarrow L$ , the embedding is *primitive* if  $L/V$  is free. In particular, a sublattice is primitive if its embedding is primitive. Two primitive embeddings  $V \hookrightarrow L$  and  $V \hookrightarrow L'$  are *isomorphic* if there is an isometry between  $L$  and  $L'$  which induces the identity on  $V$ . For a sublattice  $V$  of  $L$  we define the *orthogonal* lattice  $V^\perp := \{x \in L : b(x, y) = 0, \forall y \in V\}$ . Given two lattices  $(L_1, b_1)$  and  $(L_2, b_2)$ , their *orthogonal direct sum* is the lattice  $(L, b)$ , where  $L = L_1 \oplus L_2$  and  $b(x_1 + y_1, x_2 + y_2) = b_1(x_1, x_2) + b_2(y_1, y_2)$ , for  $x_1, x_2 \in L_1$  and  $y_1, y_2 \in L_2$ .

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2000 *Mathematics Subject Classification.* 14J28.

*Key words and phrases.* K3 surfaces, Fourier-Mukai partners.

<sup>1</sup>This result was independently proved by Hosono, Lian, Oguiso and Yau (Theorem 2.1 in [5]).

The *dual lattice* of a lattice  $(L, b)$  is  $L^\vee := \text{Hom}(L, \mathbb{Z}) \cong \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : b(x, y) \in \mathbb{Z}, \forall y \in L\}$ . Given the natural inclusion  $L \hookrightarrow L^\vee$ ,  $x \mapsto b(-, x)$ , we define the *discriminant group*  $A_L := L^\vee/L$ . The order of  $A_L$  is  $|\det L|$  (see [1], Lemma 2.1, page 12). Moreover,  $b$  induces a symmetric bilinear form  $b_L : A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$  and a corresponding quadratic form  $q_L : A_L \rightarrow \mathbb{Q}/\mathbb{Z}$  such that, when  $L$  is even,  $q_L(\bar{x}) = q(x)$  modulo  $2\mathbb{Z}$ , where  $\bar{x}$  is the image of  $x \in L^\vee$  in  $A_L$ . The elements of the triple  $(t_{(+)}, t_{(-)}, q_L)$ , where  $t_{(\pm)}$  is the multiplicity of positive/negative eigenvalues of the quadratic form on  $L \otimes \mathbb{R}$ , are invariants of the lattice  $L$ .

If  $L$  is unimodular,  $L^\vee \cong \{b(-, x) : x \in L\}$ . If  $V$  is a primitive sublattice of a unimodular lattice  $L$  such that  $b|_V$  is non-degenerate, then there is a natural isometry of groups  $\gamma : V^\vee/V \rightarrow (V^\perp)^\vee/V^\perp$ .

**1.2. K3 surfaces and  $M$ -polarizations.** A *K3 surface* is a 2-dimensional complex projective smooth variety with trivial canonical bundle and first Betti number  $b_1 = 0$ . From now on,  $X$  will be a K3 surface. The group  $H^2(X, \mathbb{Z})$  with the cup product is an even unimodular lattice and it is isomorphic to the lattice  $\Lambda := U^3 \oplus E_8(-1)^2$  (for the meaning of  $U$  and  $E_8$  see [1] page 14). The lattice  $\Lambda$  is called *K3 lattice* and it is unimodular and even.

Given the lattice  $H^2(X, \mathbb{Z})$ , the *Néron-Severi group*  $\text{NS}(X)$  is a primitive sublattice.  $T_X := \text{NS}(X)^\perp$  is the *transcendental lattice*. The rank of the Néron-Severi group  $\rho(X) := \text{rk NS}(X)$  is called the *Picard number*, and the signature of the Néron-Severi group is  $(1, \rho - 1)$ , while the one of the transcendental lattice is  $(2, 20 - \rho)$ . If  $X$  and  $Y$  are two K3 surfaces,  $f : T_X \rightarrow T_Y$  is an *Hodge isometry* if it is an isometry of lattices and the complexification of  $f$  is such that  $f_{\mathbb{C}}(\mathbb{C}\omega_X) = \mathbb{C}\omega_Y$ , where  $H^{2,0}(X) = \mathbb{C}\omega_X$  and  $H^{2,0}(Y) = \mathbb{C}\omega_Y$ . We write  $(T_X, \mathbb{C}\omega_X) \cong (T_Y, \mathbb{C}\omega_Y)$  to say that there is an Hodge isometry between the two transcendental lattices.

A *marking* for a K3 surface  $X$  is an isometry  $\varphi : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ . We write  $(X, \varphi)$  for a K3 surface  $X$  with a marking  $\varphi$ . Given  $\Lambda_{\mathbb{C}} := \Lambda \otimes \mathbb{C}$  and given  $\omega \in \Lambda_{\mathbb{C}}$  we denote by  $[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}})$  the corresponding line and we define the set  $\Omega := \{[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : \omega \cdot \bar{\omega} > 0\}$ . The image in  $\mathbb{P}(\Lambda_{\mathbb{C}})$  of the line spanned by  $\varphi_{\mathbb{C}}(\omega_X)$  belongs to  $\Omega$  and is called *period point* (or period) of the marked surface  $(X, \varphi)$ . From now on, the period point of a marked K3 surface  $(X, \varphi)$  will be indicated either by  $\mathcal{C}\varphi_{\mathbb{C}}(\omega_X)$  or by  $[\varphi_{\mathbb{C}}(\omega_X)]$ .

Given two K3 surfaces  $X$  and  $Y$ , we say that they are *Fourier-Mukai-partners* (or FM-partners) if there is an equivalence between the bounded derived categories of coherent sheaves  $\text{D}_{\text{coh}}^b(X)$  and  $\text{D}_{\text{coh}}^b(Y)$ . By results due to Mukai and Orlov, this is equivalent to say that there is an Hodge isometry  $(T_X, \mathbb{C}\omega_X) \rightarrow (T_Y, \mathbb{C}\omega_Y)$ . We define  $\text{FM}(X)$  to be the set of the isomorphism classes of the FM-partners of  $X$ .

Let  $M$  be a primitive sublattice of  $\Lambda$  with signature  $(1, t)$ . A K3 surface  $X$  with a marking  $\varphi : H^2(X, \mathbb{Z}) \rightarrow \Lambda$  is a *marked  $M$ -polarized K3 surface* if  $\varphi^{-1}(M) \subseteq \text{NS}(X)$ . A K3 surface is  *$M$ -polarizable* if there is a marking  $\varphi$  such that  $(X, \varphi)$  is a marked  $M$ -polarized K3 surface. Two marked and  $M$ -polarized surfaces  $(X, \varphi)$  and  $(X', \varphi')$  are isomorphic if there is an isomorphism  $\psi : X \rightarrow X'$  such that  $\varphi' = \varphi \circ \psi^*$ . From now on, we will consider the case of lattices  $M := \langle h \rangle$ , with  $h^2 = 2d$  and  $d > 0$ . The pair  $(X, h)$ , where  $X$  is a K3 surface and  $h \in \text{NS}(X)$ , with  $h^2 = 2d$ , means a K3 surface with a polarization of degree  $2d$ .

## 2. FM-PARTNERS OF A K3 SURFACE WITH $\rho = 1$ AND ASSOCIATED MUKAI VECTORS

In this section we want to describe the Mukai vectors of the moduli spaces associated to the  $M$ -polarized FM-partners of a K3 surface  $X$  with Picard number 1. By Orlov's results ([14]),  $q = |\text{FM}(X)|$  is the same as the number of non-isomorphic compact 2-dimensional fine moduli spaces of stable sheaves on  $X$ . Obviously, on a K3 surface with Picard number 1 and  $\text{NS}(X) = \langle h \rangle$  there is only one  $\langle h \rangle$ -polarization of degree  $h^2 = 2d$ . So the concept of FM-partner and the concept of  $M$ -polarized FM-partner coincide. If  $M = \langle h \rangle$  we are sure, by Orlov, that if we find

$q$  non-isomorphic moduli spaces, then these are representatives of all the isomorphism classes of  $M$ -polarized FM-partners of  $X$ .

We recall briefly the counting formula for the isomorphism classes of FM-partners of a given K3 surface. Given a lattice  $S$ , the *genus* of  $S$  is the set  $\mathcal{G}(S)$  of all the isometry classes of lattices  $S'$  such that  $A_S \cong A_{S'}$  and the signature of  $S'$  is equal to the one of  $S$ .

Let  $T_X$  be the transcendental lattice of an abelian surface or of a K3 surface  $X$  with period  $\mathbb{C}\omega_X$ . We can define the group

$$G := O_{\text{Hodge}}(T_X, \mathbb{C}\omega_X) = \{g \in O(T_X) : g(\mathbb{C}\omega_X) = \mathbb{C}\omega_X\}.$$

We know (see [2] Theorem 1.1, page 128), that the genus of a lattice, with fixed rank and discriminant, is finite. The map  $O(S) \rightarrow O(A_S)$  defines an action of  $O(S)$  on  $O(A_S)$ . On the other hand, taken  $g \in G$ , and given a marking  $\varphi$  for  $X$ ,  $\varphi \circ g \circ \varphi^{-1}$  induces an isometry on the lattice  $T := \varphi(T_X)$ , thus  $\varphi$  defines a homomorphism  $G \hookrightarrow O(T)$ . The composition of this map and the map  $O(T) \rightarrow O(A_T)$  gives an action of  $G$  on  $O(A_T) \cong O(A_S)$ .

**Theorem 2.1.** [4, Theorem 2.3]. *Let  $X$  be a K3 surface and let  $\mathcal{G}(\text{NS}(X)) = \mathcal{G}(S) = \{S_1, \dots, S_m\}$ . Then*

$$|FM(X)| = \sum_{j=1}^m |O(S_j) \backslash O(A_{S_j}) / G|,$$

where the actions of the groups  $G$  and  $O(S_j)$  are defined as before.

The following corollary (which is Theorem 1.10 in [13]) determines the number  $q$  of FM-partners of a surface with Picard number 1.

**Corollary 2.2.** *Let  $X$  be a K3 surface with  $\rho(X) = 1$  and such that  $\text{NS}(X) = \langle h \rangle$ , with  $h^2 = 2d$ .*

(i) *The group  $O(A_S)$  is trivial if  $d = 1$  while, if  $d > 1$ ,  $O(A_S) \cong (\mathbb{Z}/2\mathbb{Z})^{p(d)}$ , where  $p(d)$  is the number of distinct primes  $q$  such that  $q|d$ . In particular, if  $d \geq 2$ , then  $|O(A_S)| = 2^{p(d)}$ .*

(ii) *For all markings  $\varphi$  of  $X$ , the image of  $H_{X,\varphi} := \{\varphi \circ g \circ \varphi^{-1} : g \in G\} \subseteq O(T)$  in  $O(A_T)$  by the map  $O(T) \rightarrow O(A_T)$  is  $\{\pm i\bar{d}\}$ .*

*In particular,  $|FM(X)| = 2^{p(d)-1}$ , where now we set  $p(1) = 1$ .*

Assertion (i) is known and it can also be found in [15] (Lemma 3.6.1).

Using the notation of [10], we put  $H^*(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ . Given  $\alpha := (\alpha_1, \alpha_2, \alpha_3)$  and  $\beta := (\beta_1, \beta_2, \beta_3)$  in  $H^*(X, \mathbb{Z})$ , using the cup product we define the bilinear form

$$\alpha \cdot \beta := -\alpha_1 \cup \beta_3 + \alpha_2 \cup \beta_2 - \alpha_3 \cup \beta_1.$$

From now on, depending on the context,  $\alpha \cdot \beta$  will mean the bilinear form defined above or the cup product on  $H^2(X, \mathbb{Z})$ .

We give to  $H^*(X, \mathbb{Z})$  an Hodge structure considering

$$\begin{aligned} H^*(X, \mathbb{C})^{2,0} &:= H^{2,0}(X), \\ H^*(X, \mathbb{C})^{0,2} &:= H^{0,2}(X), \\ H^*(X, \mathbb{C})^{1,1} &:= H^0(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbb{C}). \end{aligned}$$

$\tilde{H}(X, \mathbb{Z})$  is the group  $H^*(X, \mathbb{Z})$  with the bilinear form and the Hodge structure defined before.

For  $v = (r, h, s) \in \tilde{H}(X, \mathbb{Z})$  with  $r \in H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ ,  $s \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$  and  $h \in H^2(X, \mathbb{Z})$ ,  $M(v)$  is the moduli space of stable sheaves  $E$  on  $X$  such that  $\text{rk} E = r$ ,  $c_1(E) = h$  and  $s = c_1(E)^2/2 - c_2(E) + r$ . If the stability is defined with respect to  $A \in H^2(X, \mathbb{Z})$  we write  $M_A(v)$ . The vector  $v$  is *isotropic* if  $v \cdot v = 0$ . The vector  $v \in \tilde{H}(X, \mathbb{Z})$  is *primitive* if  $\tilde{H}(X, \mathbb{Z})/\mathbb{Z}v$  is free.

As we have observed, the results of Orlov in [14] imply that each FM-partner of  $X$  is isomorphic to an  $M_h(v)$ . We determine a set of Mukai vectors which corresponds bijectively to the isomorphism classes of the FM-partners of  $X$  in  $FM(X)$ . First of all, we recall the following theorem due to Mukai ([10]).

**Theorem 2.3.** [10, Theorem 1.5 3]. *If  $X$  is a K3 surface,  $v = (r, h, s)$  is an isotropic vector in  $\tilde{H}^{1,1}(X, \mathbb{Z}) = H^*(X, \mathbb{C})^{1,1} \cap H^*(X, \mathbb{Z})$  and  $M_A(v)$  is non-empty and compact, then there is an isometry  $\varphi : v^\perp / \mathbb{Z}v \rightarrow H^2(M_A(v), \mathbb{Z})$  which respects the Hodge structure.*

If  $\text{NS}(X) \cong \mathbb{Z}h$  with  $h^2 = 2d = 2p_1^{e_1} \dots p_m^{e_m}$ , where  $k \geq 0$ ,  $e_i \geq 1$  and  $p_i$  primes with  $p_i \neq p_j$  if  $i \neq j$ , then we consider the Mukai vectors

$$v_J^I = v_{j_{s+1}, \dots, j_m}^{j_1, \dots, j_s} = (p_{j_1}^{e_{j_1}} \dots p_{j_s}^{e_{j_s}}, h, p_{j_{s+1}}^{e_{j_{s+1}}} \dots p_{j_m}^{e_{j_m}}),$$

where  $I = \{j_1, \dots, j_s\}$  and  $J = \{j_{s+1}, \dots, j_m\}$  are a partition of  $\{1, \dots, m\}$  such that  $I \amalg J = \{1, \dots, m\}$ . The following theorem shows how to determine  $|FM(X)|$  of them corresponding to non-isomorphic moduli spaces of stable sheaves.

**Theorem 2.4.** *Let  $X$  be a K3 surface with  $\text{NS}(X) = \mathbb{Z}h$  such that  $h^2 = 2d = 2p_1^{e_1} \dots p_m^{e_m}$ . Then, for all  $v_J^I$  as above,  $M_h(v_J^I)$  is a 2-dimensional compact fine moduli space of stable sheaves on  $X$ . Moreover, if  $M_h(v_{J_1}^{I_1}) \cong M_h(v_{J_2}^{I_2})$ , then  $v_{J_1}^{I_1} = v_{J_2}^{I_2}$  or  $v_{J_2}^{I_2} = (s_1, h, r_1)$ , with  $v_{J_1}^{I_1} = (r_1, h, s_1)$ , where the multindexes  $I_k$  and  $J_k$  vary over all the partitions of  $\{1, \dots, m\}$ .*

*Proof.* The vectors  $v_J^I$  are all isotropic and they are primitive in  $\tilde{H}(X, \mathbb{Z})$ , so, by Theorem 5.4 in [10]  $M_h(v_J^I)$  is non-empty. Moreover the hypothesis of Theorem 4.1 in [10], are satisfied and so the moduli spaces are compact. By Corollary 0.2 in [11] they are 2-dimensional, while they are fine by the results in the appendix of [10].

If  $m = 0$  or  $m = 1$  then, by Corollary 2.2, we have only one moduli space with respectively  $v = (1, h, 1)$  in the first case and  $v = (1, h, p^e)$  in the second case.

Otherwise we must prove that if

$$v_1 = (r_1, h, s_1) = v_{J_1}^{I_1} \neq v_{J_2}^{I_2} = (r_2, h, s_2) = v_2,$$

with  $v_2 \neq (s_1, h, r_1)$ , then

$$M_h(v_1) \not\cong M_h(v_2)$$

But by Theorem 2.3 and Torelli theorem, if we put

$$M_1 := v_1^\perp / \mathbb{Z}v_1 \quad \text{and} \quad M_2 := v_2^\perp / \mathbb{Z}v_2$$

then it suffices to show that there are no Hodge isometries between  $M_1$  and  $M_2$ . Obviously, it suffices to show that there are no Hodge isometries between the transcendental lattices which lifts to an isometry of the second cohomology groups.

By definition, a representative of a class in  $M_i$  ( $i = 1, 2$ ) is a vector  $(a, b, c)$  such that  $bh = as_i + cr_i$ , hence

$$bh \equiv as_i \pmod{r_i},$$

for  $i = 1, 2$ . From now on we will write  $(a, b, c)$  for the equivalence class or for a representative of the class. In fact, all the arguments we are going to propose are independent from the choice of a representative.

The Hodge structures on  $M_1$  and  $M_2$  are induced by the ones defined on  $\tilde{H}(X, \mathbb{Z})$ , so, up to an isometry, we identify  $\text{NS}(M_h(v_1))$  and  $\text{NS}(M_h(v_2))$  with

$$S_1 := \langle (0, h, 2s_1) \rangle \subset M_1 \quad \text{and} \quad S_2 := \langle (0, h, 2s_2) \rangle \subset M_2$$

respectively.

Now, we can describe the transcendental lattices  $T_1 := S_1^\perp$  and  $T_2 := S_2^\perp$  of  $M_h(v_1)$  and  $M_h(v_2)$  respectively.

If  $(a, b, c) \cdot (0, h, 2s_1) = 0$  then  $bh \equiv 0 \pmod{r_1}$ . Indeed, let us suppose that  $bh \equiv K \pmod{r_1}$  where  $K \not\equiv 0 \pmod{r_1}$ . Then, by simple calculations, we obtain

$$(a, b, c) = (L, 0, H) + \left( 0, n, \frac{n \cdot h - K}{r_1} \right)$$

as equivalence classes. Here  $n = b - kh$ , for a particular  $k \in \mathbb{Z}$ ,  $bh \equiv Ls_1 \pmod{r_1}$  and  $H$  is an integer. But now

$$\begin{aligned} 0 &= (a, b, c) \cdot (0, h, 2s_1) = -2Ls_1 + nh = -2Ls_1 + (b - kh)h = \\ &= -2bh + 2wr_1 + bh - kh^2, \end{aligned}$$

with  $w \in \mathbb{Z}$ . So

$$bh \equiv 0 \pmod{r_1}.$$

This is a contradiction. By these remarks and simple calculations, a class  $y$  in  $T_1$ , as an element of the quotient  $M_1$ , has representative  $(0, n, nh/r_1)$ . But  $(0, n, nh/r_1) \cdot (0, h, 2s_1) = nh = 0$ . So  $y = (0, n, 0)$  and

$$T_1 = \{(0, n, 0) : n \in T_X\}.$$

Analogously we have

$$T_2 = \{(0, n, 0) : n \in T_X\}.$$

By Lemma 4.1 in [13] (see also point (ii) of Corollary 2.2), if  $f : (T_1, \mathbb{C}\omega_1) \rightarrow (T_2, \mathbb{C}\omega_2)$  is a Hodge isometry, then all the Hodge isometries from  $T_1$  into  $T_2$  are  $f$  and  $-f$ . But in this case  $M_1$  and  $M_2$  inherit their Hodge structure from  $\tilde{H}(X, \mathbb{Z})$ . Hence the two Hodge isometries  $f, g : T_1 \rightarrow T_2$  are

$$(0, n, 0) \xrightarrow{f} (0, n, 0) \quad \text{or} \quad (0, n, 0) \xrightarrow{g} (0, -n, 0).$$

Let us show that  $f$  cannot be lifted to an isometry from  $M_1$  into  $M_2$ . Equivalently, this means that there are no isomorphisms between  $M_h(v_1)$  and  $M_h(v_2)$  which induces  $f$ .

We start by observing that, if  $(a, b, c) \in M_i$  with  $i = 1, 2$ , then

$$(a, b, c) \cdot (0, h, 2s_i) \equiv -bh \pmod{r_i}.$$

Indeed, if  $bh \equiv K \pmod{r_i}$  then  $a \equiv L \pmod{r_i}$  and so  $(a, b, c) = (L, 0, H) + (0, n, \frac{n \cdot h - K}{r_i})$  with  $n$  and  $H$  as before. So,  $(a, b, c) \cdot (0, h, 2s_i) = -2bh + 2wr_i + bh - kh^2 \equiv -bh \pmod{r_i}$  and  $nh \equiv bh \pmod{r_i}$ .

Now let us suppose that there is an isometry  $\varphi : M_1 \rightarrow M_2$  which induces  $f$ . We can prove that there is  $(a, b, c) \in M_1$ , with  $bh \equiv 0 \pmod{r_1}$ , such that  $\varphi(a, b, c) = (d, e, f) \in M_2$  with  $eh \not\equiv 0 \pmod{r_2}$ . First of all, by our hypotheses about  $r_1$  and  $r_2$ , we can suppose that there is a prime  $p$  which divides  $r_2$  but which does not divide  $r_1$  (otherwise we can change the roles of  $M_1$  and  $M_2$  in the following argument). By Theorem 1.14.4 in [12], there is an isometry

$$\psi : H^2(X, \mathbb{Z}) \longrightarrow U^3 \oplus E_8(-1)^2 = \Lambda$$

such that  $k_1 := \psi(h) = (1, d, 0, \dots, 0)$ , where  $h^2 = 2d$ . Let  $k_2 := (0, r_1, 0, \dots, 0)$ . Now  $k_1 \cdot k_2 = r_1$  and we can take  $n := \psi^{-1}(k_2)$ . Obviously, the vector  $(0, n, n \cdot h/r_1) \in M_1$  is such that  $n \cdot h \equiv 0 \pmod{r_1}$ . Let us suppose that  $\varphi((0, n, n \cdot h/r_1)) = (d, e, f)$  with  $e \cdot h \equiv 0 \pmod{r_2}$ . By the previous remark, this is equivalent to say that

$$\varphi \left( \left( 0, n, \frac{n \cdot h}{r_1} \right) \right) = \left( 0, m, \frac{m \cdot h}{r_2} \right),$$

for a given  $m \in H^2(X, \mathbb{Z})$ .

Since  $\text{rk}S_1 = \text{rk}S_2 = 1$ , either  $\varphi((0, h, 2s_1)) = (0, h, 2s_2)$  or  $\varphi((0, h, 2s_1)) = -(0, h, 2s_2)$ . In particular, if  $\varphi$  correspond to case (1) (the same argument holds if  $\varphi$  is as in case (2)), then

$$\begin{aligned} n \cdot h &= \left( 0, n, \frac{n \cdot h}{r_1} \right) \cdot (0, h, 2s_1) = \varphi \left( \left( 0, n, \frac{n \cdot h}{r_1} \right) \cdot (0, h, 2s_1) \right) = \\ &= \left( 0, m, \frac{m \cdot h}{r_2} \right) \cdot (0, h, 2s_2) = m \cdot h. \end{aligned}$$

In particular,  $m \cdot h = n \cdot h = r_1$  which is not divisible by  $r_2$ . This gives a contradiction and thus  $eh \not\equiv 0 \pmod{r_2}$ .

The previous remarks show that if

$$(a, b, c) = \left(0, n, \frac{nh}{r_1}\right)$$

then

$$\varphi((a, b, c)) = (d, e, f) = (L, 0, H) + \left(0, m, \frac{m \cdot h - K}{r_2}\right),$$

with  $L \not\equiv 0 \pmod{r_2}$ . Let us take  $(0, N, 0) \in T_1$  and

$$\varphi(0, N, 0) = f(0, N, 0) = (0, N, 0) \in T_2.$$

Then

$$\begin{aligned} (*) \quad nN &= \left(0, n, \frac{nh}{r_1}\right) \cdot (0, N, 0) = \\ &= \left[ (L, 0, H) + \left(0, m, \frac{m \cdot h - K}{r_i}\right) \right] \cdot (0, N, 0) = mN. \end{aligned}$$

Because  $H^2(X, \mathbb{Z})$  is unimodular and (\*) is true for every  $N \in T_X$ , we have  $m - n = kh \in \text{NS}(X)$ , where  $k \in \mathbb{Z}$ . But now  $nh = \left(0, n, \frac{nh}{r_1}\right) \cdot (0, h, 2s_1) = [(L, 0, H) + \left(0, m, \frac{m \cdot h - K}{r_i}\right)] \cdot (0, h, 2s_2) = mh - 2Ls_2 = nh + kh^2 - 2Ls_2 = nh + 2kr_2s_2 - 2Ls_2$ . So  $L \equiv 0 \pmod{r_2}$  which is contradictory.

Repeating the same arguments for  $g$ , we see that neither  $f$  nor  $g$  lifts to an isometry of the second cohomology groups. So, by Torelli Theorem,  $M_h(v_1) \not\cong M_h(v_2)$ .  $\square$

### 3. GENUS AND POLARIZATIONS WHEN $\rho = 2$

In this paragraph we are interested in the number of non isomorphic FM-partners of K3 surfaces with a given polarization and Picard number 2.

Our main result is Theorem 3.3. First of all, we recall the following lemma which is an easy corollary of Nikulin's Theorem 1.14.2 in [12] and whose hypotheses are trivially verified if  $\rho = 2$ .

**Lemma 3.1.** *Let  $L$  be an even unimodular lattice and let  $T_1$  and  $T_2$  be two even sublattice with the same signature  $(t_{(+)}, t_{(-)})$ , where  $t_{(+)} > 0$  and  $t_{(-)} > 0$ . Let the corresponding discriminant groups  $(A_{T_1}, q_{T_1})$  and  $(A_{T_2}, q_{T_2})$  be isometric and let  $\text{rk}T_1 \geq 2 + \ell(A_{T_1})$ , where  $\ell(A_{T_1})$  is the minimal number of generators of  $A_{T_1}$ . Then  $T_1 \cong T_2$ .*

We prove the following lemma.

**Lemma 3.2.** *Let  $L_{d,n}$  be the lattice  $(\mathbb{Z}^2, M_{d,n})$ , where*

$$M_{d,n} := \begin{pmatrix} 2d & n \\ n & 0 \end{pmatrix},$$

with  $d$  and  $n$  positive integers such that  $(2d, n) = 1$ . Then

- (i) the discriminant group  $A_{L_{d,n}}$  is cyclic;
- (ii) if  $d_1, d_2, n_1$  and  $n_2$  are positive integers such that  $(2d_1, n_1) = (2d_2, n_2) = 1$  then  $A_{L_{d_1, n_1}} \cong A_{L_{d_2, n_2}}$  if and only if
  - (a.1)  $n_1 = n_2$ ;
  - (b.1) there is an integer  $\alpha$  such that  $(\alpha, n) = 1$  and  $d_1 \alpha^2 \equiv d_2 \pmod{n^2}$ ;
- (iii) if  $L_{d_1, n} \cong L_{d_2, n}$  then one of the following conditions holds
  - (a.2)  $d_1 \equiv d_2 \pmod{n}$ ;
  - (b.2)  $d_1 d_2 \equiv 1 \pmod{n}$ .

*Proof.* Let  $e_{d,n} = (1, 0)^t$  and  $f_{d,n} = (0, 1)^t$  be generators of the lattice  $L_{d,n}$ . Under the hypothesis  $(2d, n) = 1$ , (i) follows immediately because

$$A_{L_{d,n}} := L_{d,n}^\vee / L_{d,n}$$

has order  $|\det M_{d,n}| = n^2$  and it is cyclic with generator

$$\bar{f}_{d,n} := \frac{ne_{d,n} - 2df_{d,n}}{n^2}.$$

Indeed,

$$L_{d,n}^\vee = \left\langle \frac{ne_{d,n} - 2df_{d,n}}{n^2}, \frac{f_{d,n}}{n} \right\rangle$$

and  $\bar{f}_{d,n}$  has order  $n^2$  in  $A_{L_{d,n}}$ .

First we prove that the conditions (a.1) and (b.1) are necessary. The orders of  $A_{L_{d_1,n_1}}$  and  $A_{L_{d_2,n_2}}$  are  $n_1^2$  and  $n_2^2$  respectively with  $n_1, n_2 > 0$ , so  $n := n_1 = n_2$  (which is (a.1)). If  $A_{L_{d_1,n}}$  and  $A_{L_{d_2,n}}$  are isomorphic as groups, there is an integer  $\alpha$  prime with  $n$  such that the isomorphism is determined by

$$\bar{f}_{d_1,n} \mapsto \alpha \bar{f}_{d_2,n}.$$

But now

$$\bar{f}_{d_1,n}^2 = \frac{-2d_1}{n^2} \quad \bar{f}_{d_2,n}^2 = \frac{-2d_2}{n^2},$$

and if we want  $A_{L_{d_1,n}}$  and  $A_{L_{d_2,n}}$  to be isometric as lattices, we must require

$$\frac{-2d_1}{n^2} \equiv \alpha^2 \frac{-2d_2}{n^2} \pmod{2}.$$

This is true if and only if

$$d_1 \equiv \alpha^2 d_2 \pmod{n^2}.$$

So the necessity of condition (b.1) is proved. In the same way it follows that (a.1) and (b.1) are also sufficient.

Let us consider point (iii). The lattices  $L_{d_1,n}$  and  $L_{d_2,n}$  are isometric if and only if there is a matrix  $A \in \text{GL}(2, \mathbb{Z})$  such that

$$(*) \quad A^t M_{d_1,n} A = M_{d_2,n}.$$

Let  $L_{d_1,n}$  and  $L_{d_2,n}$  be isometric and let

$$A := \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Then from (\*) we obtain the two relations

- (1)  $d_2 = x^2 d_1 + xzn$ ;
- (2)  $2y(yd_1 + tn) = 0$ .

By (2) we have only two possibilities: either  $y = 0$  or  $yd_1 = -tn$ . Let  $y = 0$ . From the relation

$$1 = |\det(A)| = |xt - yz| = |xt|$$

it follows that  $x = \pm 1$  and so, from (1), we have  $d_1 \equiv d_2 \pmod{n}$ , which is condition (a.2).

Let us consider the case  $yd_1 = -tn$ . We know that  $(d_1, n) = 1$  and hence  $y = cn$  and  $t = -cd_1$ , with  $c \in \mathbb{Z}$ . From

$$1 = |\det(A)| = |-cxd_1 - cnz|$$

it follows that  $c = \pm 1$ . We suppose  $c = 1$  (if  $c = -1$  then the same arguments work by simple changes of signs). Multiplying both members of relation (1) by  $d_1$  we have

$$d_2 d_1 \equiv x^2 d_1^2 \pmod{n}.$$

But we know that  $\pm 1 = \det(A) = -xd_1 - nz$  and so  $-xd_1 \equiv \pm 1 \pmod{n}$ . Thus

$$1 \equiv x^2 d_1^2 \pmod{n}$$

and from this we obtain (b.2). □

Now we can prove the following theorem (note that point (iii) and (v) are exactly Theorem 1.7 in [13]).

**Theorem 3.3.** *Let  $N$  and  $d$  be positive integers. Then there are  $N$  K3 surfaces  $X_1, \dots, X_N$  with Picard number  $\rho = 2$  such that*

- (i)  $X_i$  is elliptic, for every  $i \in \{1, \dots, N\}$ ;
- (ii) there is  $i \in \{1, \dots, N\}$  such that  $X_i$  has a polarization of degree  $2d$ ;
- (iii)  $\text{NS}(X_i) \not\cong \text{NS}(X_j)$  if  $i \neq j$ , where  $i, j \in \{1, \dots, N\}$ ;
- (iv)  $|\text{defNS}(X_i)|$  is a square, for every  $i \in \{1, \dots, N\}$ ;
- (v) there is an Hodge isometry between  $(T_{X_i}, \mathbb{C}\omega_{X_i})$  and  $(T_{X_j}, \mathbb{C}\omega_{X_j})$ , for all  $i, j \in \{1, \dots, N\}$ .

In particular,  $X_i$  and  $X_j$  are non-isomorphic FM-partners, for all  $i, j \in \{1, \dots, N\}$ .

*Proof.* The surjectivity of the period map for K3 surfaces implies that, given a sublattice  $S$  of  $\Lambda$  with rank 2 and signature  $(1, 1)$ , there is at least one K3 surface  $X$  such that its transcendental lattice  $T_X$  is isometric to  $T := S^\perp$ .

So the theorem follows if we can show that, for an arbitrary integer  $N$ , there are at least  $N$  sublattices of  $\Lambda$  with rank 2, signature  $(1, 1)$  and representing zero which are non-isometric but whose orthogonal lattices are isometric in  $\Lambda$ .

Let us consider in  $U \oplus U \hookrightarrow \Lambda$  the following sublattices

$$S_{d,n} := \left\langle \begin{pmatrix} 1 \\ q \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ n \\ 1 \\ 0 \end{pmatrix} \right\rangle,$$

with  $(2q, n) = 1$  and  $n > 0$ .

We can observe that, when  $n$  and  $d$  vary, the lattices  $S_{q,n}$  are primitive in  $\Lambda$  and the matrices associated to their quadratic forms are exactly the  $M_{q,n}$ . Since  $M_{q,n}$  has negative determinant, the lattice has signature  $(1, 1)$ . Moreover,  $S_{q,n}$  represents zero

Let  $n > 2$  be a prime number such that  $n > d^2 N^4$ . We choose  $d_1 := d$ ,  $d_2 := d^2, \dots, d_N = dN^2$ . By definition, there is an integer  $\alpha_i$  such that  $(\alpha_i, n) = 1$  and

$$\alpha_i^2 d_1 \equiv d_i \pmod{n^2},$$

for every  $i \in \{1, \dots, N\}$ . Thus the hypotheses (b.1) of Lemma 3.2 are satisfied and by point (ii) of the same lemma,

$$AS_{d_1,n} \cong AS_{d_i,n} \cong AS_{d_j,n},$$

where  $i, j \in \{1, \dots, N\}$ . By Lemma 3.2, there are isometries

$$\psi_i : S_{d_1,n}^\perp \rightarrow S_{d_i,n}^\perp,$$

with  $i \in \{2, \dots, N\}$ . Now let  $(X_1, \varphi_1)$  be a marked K3 surface associated to the lattice  $S_{d_1,n}$ . By the surjectivity of the period map we can consider the marked K3 surfaces  $(X_i, \varphi_i)$ , with  $i \in \{2, \dots, N\}$ , such that

- (1)  $\varphi_{i,\mathbb{C}}(\mathbb{C}\omega_{X_i}) = \psi_{i,\mathbb{C}}(\varphi_{1,\mathbb{C}}(\mathbb{C}\omega_{X_1}))$ ;
- (2)  $\varphi_i(\text{NS}(X_i)) = S_{d_i,n}$ ;
- (3)  $\varphi_i(T_{X_i}) = S_{d_i,n}^\perp$ .

Obviously, the surfaces  $X_i$  are FM-partners of  $X_1$ .

Now we show that, when  $i \neq j$ ,

$$S_{d_i,n} \not\cong S_{d_j,n}.$$

First of all we know that, obviously,  $d_j \not\equiv d_i \pmod{n}$  if  $i \neq j$ . On the other hand,

$$d_i d_j < d^2 N^4 < n,$$

so

$$1 \not\equiv d_i d_j \pmod{n}.$$

Hence, by point (iii) of Lemma 3.2, the lattices can not be isometric. The K3 surfaces  $X_1, \dots, X_N$  are obviously elliptic and the discriminant of their Néron-Severi group is a square. Moreover  $X_1$  has a polarization of degree  $2d$ .

This shows that it is possible to find  $N$  K3 surfaces which satisfy the hypotheses of the theorem.  $\square$

The previous theorem gives a new proof of the following result due to Oguiso ([13]).

**Corollary 3.4.** [13, Theorem 1.7]. *Let  $N$  be a natural number. Then there are  $N$  K3 surfaces  $X_1, \dots, X_N$  with Picard number  $\rho = 2$  such that*

- (i)  $\text{NS}(X_i) \not\cong \text{NS}(X_j)$  if  $i \neq j$ , where  $i, j \in \{1, \dots, N\}$ ;
- (ii) there is an Hodge isometry between  $(T_{X_i}, \mathbb{C}\omega_{X_i})$  and  $(T_{X_j}, \mathbb{C}\omega_{X_j})$ , for all  $i, j \in \{1, \dots, N\}$ .

**Remark 3.1.** The proof proposed by Oguiso in [13] is based on deep results in number theory. In particular, it uses a result of Iwaniec [7] about the existence of infinitely many integers of type  $4n^2 + 1$  which are product of two not necessarily distinct primes. Theorem 3.3 gives an elementary proof of Theorem 1.7 in [13] entirely based on simple remarks about lattices and quadratic forms.

Lemma 3.1 is true also when  $L = U \oplus U \oplus U$ . The period map is onto also for abelian surfaces (see [16]). Thus, using the lattices  $S_{d,n}$  described before, the following proposition (similar to a result given in [6]) can be proved with the same techniques.

**Proposition 3.5.** *Let  $N$  and  $d$  be positive integers. Then there are  $N$  abelian surfaces  $X_1, \dots, X_N$  with Picard number  $\rho = 2$  such that*

- (i)  $\text{NS}(X_i) \not\cong \text{NS}(X_j)$  if  $i \neq j$ , with  $i, j \in \{1, \dots, N\}$ ;
- (ii) there is  $i \in \{1, \dots, N\}$  such that  $X_i$  has a polarization of degree  $2d$ ;
- (iii) there is an Hodge isometry between  $(T_{X_i}, \mathbb{C}\omega_{X_i})$  and  $(T_{X_j}, \mathbb{C}\omega_{X_j})$ , for all  $i, j \in \{1, \dots, N\}$ .

The following easy remark shows that it is possible to obtain an arbitrarily large number of  $M$ -polarizations on a K3 surface, for certain  $M$ .

**Remark 3.2.** Let  $N$  be a natural number. Then there are a primitive sublattice  $M$  of  $\Lambda$  with signature  $(1, 0)$  and a K3 surface  $X$  with  $\rho(X) = 2$  such that  $X$  has at least  $N$  non-isomorphic  $M$ -polarizations. In particular  $X$  has at least  $N$  non-isomorphic  $M$ -polarized FM-partners.

In fact, let  $S \cong U$ , where  $U$  is, as usual, the hyperbolic lattice. Then, by the surjectivity of the period map, there is a K3 surface  $X$  such that  $\text{NS}(X) \cong S$ .

Let  $d$  be a natural number with  $d = p_1^{e_1} \dots p_n^{e_n}$ . In  $S$  there are  $2^{p(d)-1}$  primitive vectors with autointersection  $2d$ . Indeed they are all the vectors of type

$$f_J^I := (p_{j_1}^{e_{j_1}} \dots p_{j_s}^{e_{j_s}}, p_{j_{s+1}}^{e_{j_{s+1}}} \dots p_{j_n}^{e_{j_n}}),$$

for  $I$  and  $J$  that vary in all possible partitions  $I \amalg J = \{1, \dots, n\}$ .

The group  $O(U)$  has only four elements (i.e.  $\pm id$ , the exchange of the vectors of the base and the composition of this map with  $-id$ ). So it is easy to verify that all these polarizations are not isomorphic. Choosing  $d$  to be divisible by a sufficiently large number of distinct primes, we can find at least  $N$  non-isomorphic  $M$ -polarizations. The last assertion follows from Lemma 3.1.

**Acknowledgements.** The author would like to express his thanks to Professor Bert van Geemen for his suggestions and helpful discussions.

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