

A MOSER-TYPE INEQUALITY IN LORENTZ-SOBOLEV SPACES FOR UNBOUNDED DOMAINS IN \mathbb{R}^N

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ABSTRACT. We derive a Trudinger-Moser type embedding for the Lorentz-Sobolev space $W_0^1 L^{N,q}(\Omega)$, where $\Omega \subseteq \mathbb{R}^N$ is any sufficiently smooth domain and in particular for $\Omega = \mathbb{R}^N$. Precisely, we first prove that the corresponding inequality is *domain independent* and then, by constructing explicit concentrating sequences à la Moser, we establish that the embedding inequality is *sharp*.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a sufficiently smooth bounded domain and let $W_0^{1,p}(\Omega)$ be the usual Sobolev space obtained by completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_p$. The Sobolev embedding theorem reads as follows (see e.g. [3]):

$$(1) \quad W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{Np}{N-p}}(\Omega), & 1 \leq p < N \\ L^\infty(\Omega), & p > N \end{cases}$$

where $\frac{Np}{N-p} =: p^*$, is the Sobolev critical exponent. In one hand, we have that the embedding (1) is sharp from two different points of view: in the following inequality,

$$(2) \quad \|u\|_{p^*} \leq S_p \|\nabla u\|_p, \quad 1 \leq p < N$$

no larger exponent can replace p^* as well as the best constant S_p is explicitly known (domain independent and achieved just in the whole \mathbb{R}^N , see e.g. [27]). On the other hand, the Lebesgue space setting is not optimal, in the sense that the target space in (1) can be strictly smaller than $L^{p^*}(\Omega)$. Indeed, it was proved in [4] that (2) can be

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improved in the context of Lorentz spaces $(L^{p,q}(\Omega), \|\cdot\|_{p,q})$ (which we recall in Section 2) by means of the following inequality:

$$(3) \quad \|u\|_{p^*,p} \leq C \|\nabla u\|_p$$

where $L^{p^*,p} \subset L^{p^*,p^*} \equiv L^{p^*}$ and thus (3) is a refinement of (2).

The so-called limiting Sobolev case, occurs when $N = p$ and thus $p^* = \infty$ even though $W_0^{1,N}(\Omega) \hookrightarrow L^\infty(\Omega)$ holds true just in dimension $N = 1$. Indeed, for $p = N \geq 2$, a famous result obtained independently by Pohožaev [23] and Trudinger [28] states, in the sharp version due to Moser [22], the following:

$$(4) \quad \sup_{\|\nabla u\|_N \leq 1} \int_{\Omega} \left(e^{\alpha|u(x)|^{\frac{N}{N-1}}} - 1 \right) dx \leq C(\alpha, N)|\Omega| \begin{cases} < \infty, & \text{if } \alpha \leq \alpha_N \\ = \infty, & \text{if } \alpha > \alpha_N \end{cases}$$

where $\alpha_N = \left(N \omega_N^{\frac{1}{N}} \right)^{\frac{N}{N-1}}$, denoting with ω_N the measure of the unit ball in \mathbb{R}^N .

This result has been extended in many directions: a generalization to higher order fractional Sobolev spaces is given in [26], an analog of (4) for higher order derivatives is established in [2] whereas a sharp form without boundary condition is obtained in [11]. Attainability is also studied in [10] where it is proved that the supremum in (4) is achieved up to the critical value α_N ; a different prove can be found in [12] together with some applications. In particular, improvability of (4) by exploiting a Lorentz space setting has been addressed in [5] and [8] (see also the appendix in [7]), results further extended in [6] by using different techniques, some of which we are going to borrow here, where the authors prove the following sharp result:

$$(5) \quad \sup_{\|\nabla u\|_{N,q} \leq 1} \int_{\Omega} \left(e^{\alpha|u(x)|^{\frac{q}{q-1}}} - 1 \right) dx \leq C(\alpha, N, q)|\Omega| \begin{cases} < \infty, & \alpha \leq \alpha_{N,q} \\ = \infty, & \alpha > \alpha_{N,q} \end{cases}$$

where $1 < q < \infty$ and $\alpha_{N,q} = \left(N \omega_N^{\frac{1}{N}} \right)^{\frac{q}{q-1}}$. Clearly, (5) reduces to (4) when $q = N$ but does improve the maximal growth in (4) as long as $1 < q < N$ (in this case, see also [16] for a different proof of (5)).

Clearly, the right hand sides in both inequalities (4) and (5) depend on the measure $|\Omega|$ of the underlying domain and no informations can be retained as $|\Omega| \rightarrow \infty$. However,

the validity of Trudinger-type inequalities in the whole space have been investigated in [9, 13, 1] and more recently in [24, 19], where it is shown that a sharp uniform bound as in (4) is still available for any domain, provided that the norm $\|\nabla u\|_N$ is replaced by the Sobolev norm: $\|\nabla u\|_N + \|u\|_N$. Exploiting techniques introduced in [2, 7, 6] and in the same spirit of [24], here we tackle the problem of extending (5) to any sufficiently smooth domain, not necessarily with finite measure, in particular to the whole \mathbb{R}^N . Precisely, we derive, as in [24], an optimal inequality which does not depend on the domain Ω and thus valid in the whole \mathbb{R}^N . For the sake of clarity, we adopt a layered presentation by proving first the results in dimension $N = 2$, which we then extend to higher dimensions.

Our main results are the following:

Theorem 1. *Let $1 < q < \infty$. Then, there exists a positive constant $C = C(q)$ such that for any (sufficiently smooth) domain $\Omega \subseteq \mathbb{R}^2$ and for any $\alpha \leq \alpha_{2,q} = (\sqrt{4\pi})^{\frac{q}{q-1}}$, the following inequalities hold:*

$$(6) \quad \sup_{\|u\|_{1,(2,q)} \leq 1} \int_{\Omega} \left(e^{\alpha|u|^{\frac{q}{q-1}}} - 1 - \alpha|u|^{\frac{q}{q-1}} \right) dx \leq C, \quad \text{if } 2 < q < \infty$$

$$(7) \quad \sup_{\|u\|_{1,(2,q)} \leq 1} \int_{\Omega} \left(e^{\alpha|u|^{\frac{q}{q-1}}} - 1 \right) dx \leq C, \quad \text{if } 1 < q \leq 2$$

where $\|u\|_{1,(p,q)}^q := \|\nabla u\|_{p,q}^q + \|u\|_{p,q}^q$. Moreover, inequalities (6)–(7) are sharp, in the sense that for any $\alpha > \alpha_q$ the corresponding suprema become infinity.

Theorem 2. *Let $1 < q < \infty$, $N \geq 3$ and set*

$$\Phi(t) := e^t - \sum_{j=1}^{k_0} \frac{t^j}{j!}, \quad k_0 = \left\lfloor \frac{(q-1)N}{q} \right\rfloor$$

Then, there exists a constant $C = C(N, q) > 0$ such that for any (sufficiently smooth) domain $\Omega \subseteq \mathbb{R}^N$ and for any $\alpha \leq \alpha_{N,q} := (N\omega_N^{1/N})^{\frac{q}{q-1}}$ (where ω_N is the measure of the unit ball in \mathbb{R}^N) the following inequality holds:

$$(8) \quad \sup_{\|u\|_{1,(N,q)} \leq 1} \int_{\Omega} \Phi\left(\alpha|u|^{\frac{q}{q-1}}\right) dx \leq C$$

Moreover, inequality (8) is sharp: for any growth $\Phi(\alpha|u|^{\frac{q}{q-1}})$ with $\alpha > \alpha_{N,q}$, the supremum becomes infinity.

The paper is organized as follows: first, we briefly recall some preliminaries on functional rearrangements and Lorentz spaces, then in Section 3 we prove inequalities (6) and (7); a key ingredient is Adams' lemma [2]. In Section 4, we construct as in [6, 24], an explicit Moser-type concentrating sequence in a Lorentz-Sobolev setting, which we use to show that inequalities (6) and (7) are sharp. Finally, Section 5 is devoted to extending the previous results to higher dimensions, in particular we prove inequality (8), for which optimality turns out to be a delicate issue. We complete this paper with some remarks on the borderline cases $q = 1$ and $q = \infty$.

2. PRELIMINARY RESULTS

Let $\phi : \Omega \rightarrow \mathbb{R}^+$ be a measurable function. Denoting by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$ let

$$\mu_\phi(t) = |\{x \in \Omega : \phi(x) > t\}|, \quad t \geq 0$$

be the distribution function of ϕ . The *decreasing rearrangement* $\phi^*(s)$ of ϕ is defined as the distribution function of μ_ϕ , that is

$$\phi^*(s) = |\{t \in [0, +\infty) : \mu_\phi(t) > s\}| = \sup\{t > 0 : \mu_\phi(t) > s\}, \quad s \in [0, |\Omega|]$$

whereas the *spherically symmetric rearrangement* $\phi^\#(x)$ of ϕ can be defined as

$$\phi^\#(x) = \phi^*(\omega_N |x|^N), \quad x \in \Omega^\#$$

where $\Omega^\# \subset \mathbb{R}^N$ is the open ball with center in the origin which satisfies $|\Omega^\#| = |\Omega|$. (For equivalent definitions and an insight on functional rearrangement theory, we refer the reader to [20, 27, 18].)

Definition 1. Let $1 < p < \infty$, $1 \leq q < +\infty$ and ϕ a measurable function on Ω . Let

$$\bar{\phi}(t) := \frac{1}{t} \int_0^t \phi^*(s) ds, \quad t > 0$$

Then $\phi \in L^{p,q}(\Omega)$ provided that:

$$(9) \quad |||\phi|||_{p,q} := \left(\frac{q}{p} \int_0^\infty (\bar{\phi}(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

The spaces $(L^{p,q}(\Omega), ||| \cdot |||_{p,q})$, introduced by G.G. Lorentz in [21], are Banach spaces, which are reflexive for $1 < p, q < \infty$. We recall from [17] that if $1 \leq q_1 < p < q_2 < \infty$, then the following inclusions hold:

$$(10) \quad L^{p,q_1} \subset L^{p,p} \equiv L^p \subset L^{p,q_2} \subset \begin{cases} L^{p_1,q_1}(\Omega), & \text{if } p_1 < p \text{ and } |\Omega| < \infty \\ L^{p_2,q_2}(\Omega), & \text{if } p_2 > p \text{ and } |\Omega| \geq 1 \end{cases}$$

(The inclusions related to the second index are usually referred as the *Lorentz scale*.)

Besides the norm in (9) it is sometimes convenient to consider the following quantity

$$(11) \quad \|\phi\|_{p,q} = \left(\int_0^\infty \left(\phi^*(t) t^{1/p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

which is equivalent to $||| \cdot |||_{p,q}$ even though for $q > p$ it is not a norm since it does not satisfy the triangle inequality. Notice that, when $p = q$, one has $\| \cdot \|_{p,p} = \| \cdot \|_p$.

As Sobolev spaces are built up from Lebesgue spaces, similarly one introduces Lorentz-Sobolev spaces which consist of functions having weak derivatives belonging to the Lorentz space $L^{p,q}$. In particular we introduce the following

Definition 2. Let $\Omega \subseteq \mathbb{R}^N$ be a smooth domain. Assume that $1 < q < +\infty$ and let us define by completion the Lorentz-Sobolev space $W_0^1 L^{N,q}(\Omega)$, namely

$$W_0^1 L^{N,q}(\Omega) = cl \{ u \in C_0^\infty(\Omega) : \|\nabla u\|_{1,(N,q)} < \infty \}$$

where

$$(12) \quad \|u\|_{1,(N,q)}^q := \|u\|_{N,q}^q + \|\nabla u\|_{N,q}^q$$

In what follows we will use the following relation between nonnegative functions belonging to $L^1(\Omega)$:

Definition 3. Let $\Omega \subseteq \mathbb{R}^N$ be a domain, we say that ϕ is dominated by ψ , and we write $\phi \prec \psi$, if the following hold:

$$\begin{aligned} \int_0^s \phi^*(t) dt &\leq \int_0^s \psi^*(t) dt, & \text{for all } s \in [0, |\Omega|) \\ \int_0^{|\Omega|} \phi^*(t) dt &= \int_0^{|\Omega|} \psi^*(t) dt \end{aligned}$$

This relation was first introduced by Hardy, Littlewood and Pólya in [15] for vectors of \mathbb{R}^N and then extended to Lebesgue integrable functions on a finite interval. We refer to [7] for a survey on properties and characterizations of this relation. We have the following property, which is a trivial extension of [7, Corollary 2.1]:

Proposition 1. *Let ϕ, ψ two nonnegative functions in $L^1(\mathbb{R}^N)$ such that $\phi \prec \psi$. Then, for all nonnegative $\eta \in L^\infty(\mathbb{R}^N)$ and for any $M \in [0, +\infty]$ one has*

$$\int_0^M \phi^*(t)\eta^*(t) dt \leq \int_0^M \psi^*(t)\eta^*(t) dt$$

Proof. It suffices to observe that for any $M > 0$

$$\begin{aligned} \int_0^M \phi^*(t)\eta^*(t) dt &= - \int_0^M \left(\int_0^t \phi^*(s) ds \right) d\eta^*(t) + \eta^*(M) \int_0^M \phi^*(t) dt \\ &\leq - \int_0^M \left(\int_0^t \psi^*(s) ds \right) d\eta^*(t) + \eta^*(M) \int_0^M \psi^*(t) dt \\ &= \int_0^M \psi^*(t)\eta^*(t) dt \end{aligned}$$

since $\phi \prec \psi$ and η^* is non increasing. □

We briefly recall from [7] a procedure to construct a function Φ dominated by a given function ϕ . Let $u(x)$ be a measurable function in Ω ; then there exists a family $\{\mathcal{F}_s\}$, $s \in [0, |\Omega|]$ of subsets of Ω satisfying the following properties:

- (i) $|\mathcal{F}_s| = s$
- (ii) $s_1 < s_2 \Rightarrow \mathcal{F}_{s_1} \subset \mathcal{F}_{s_2}$
- (iii) $\mathcal{F}_s = \{x \in \Omega : |u(x)| > t\}$ if $s = \mu_u(t)$

Let $\phi \in L^1(\Omega)$ and let $\Phi(t)$ be the function defined by the following equality

$$(13) \quad \int_{\mathcal{F}_s} \phi(x) dx = \int_0^s \Phi(t) dt, \quad s \in [0, |\Omega|]$$

In this case we say that Φ is built from ϕ on the level sets of $|u|$ and one has $\Phi \prec \phi$.

3. A UNIFORM BOUND: THE CASE OF DIMENSION $N = 2$

The aim of this section is to prove Theorem 1. In the sequel, we set for the sake of simplicity $\alpha_q := \alpha_{2,q}$. We begin with the following

Proposition 2. *Let Ω be a smooth domain in \mathbb{R}^2 , and let $1 < q < \infty$. Then there exists a constant $C(q)$, independent of Ω , such that*

$$(14) \quad \sup_{\|u\|_{1,(2,q)} \leq 1} \int_{\Omega} \left(e^{\alpha_q |u|^{\frac{q}{q-1}}} - 1 - \alpha_q |u|^{\frac{q}{q-1}} \right) dx \leq C, \quad \text{if } 2 < q < \infty$$

$$(15) \quad \sup_{\|u\|_{1,(2,q)} \leq 1} \int_{\Omega} \left(e^{\alpha_q |u|^{\frac{q}{q-1}}} - 1 \right) dx \leq C, \quad \text{if } 1 < q \leq 2$$

We are going to prove the above proposition in several steps which we organize in a separate section.

3.1. Proof of Proposition 2. First notice that if $\partial\Omega$ is not empty, we assume it to be smooth, so that any function $u \in W_0^{1,L^{2,q}}(\Omega)$ can be extended by zero outside Ω to get a function in $W^1L^{2,q}(\mathbb{R}^2)$. Thus, it is enough to show that (14) and (15) hold with $\Omega = \mathbb{R}^2$. Without loss of generality, we may also assume $u \in C_0^\infty(\mathbb{R}^2)$. By applying symmetrization, one has

$$(16) \quad \int_{\mathbb{R}^2} \left(e^{\alpha_q |u|^{\frac{q}{q-1}}} - 1 - \alpha_q |u|^{\frac{q}{q-1}} \right) dx = \int_0^\infty \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 - \alpha_q |u^*|^{\frac{q}{q-1}} \right) ds$$

as well as

$$(17) \quad \int_{\mathbb{R}^2} \left(e^{\alpha_q |u|^{\frac{q}{q-1}}} - 1 \right) dx = \int_0^\infty \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 \right) ds$$

and we are going to estimate the right hand side in (16) and (17), respectively for $q > 2$ and $1 < q \leq 2$. Let $r_0 > 0$ to be suitably chosen, we split integrals into two parts:

$$(18) \quad \int_0^\infty \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 - \alpha_q |u^*|^{\frac{q}{q-1}} \right) ds \\ = \left(\int_0^{r_0} + \int_{r_0}^\infty \right) \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 - \alpha_q |u^*|^{\frac{q}{q-1}} \right) ds$$

and

$$(19) \quad \int_0^\infty \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 \right) ds = \left(\int_0^{r_0} + \int_{r_0}^\infty \right) \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 \right) ds$$

The second term in (18)–(19), which we rewrite as:

$$(20) \quad \int_{r_0}^{+\infty} \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 - \alpha_q |u^*|^{\frac{q}{q-1}} \right) ds = \sum_{k=2}^{\infty} \int_{r_0}^{\infty} \frac{\alpha_q^k |u^*|^{\frac{kq}{q-1}}}{k!}$$

$$\int_{r_0}^{+\infty} \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 \right) ds = \sum_{k=1}^{\infty} \int_{r_0}^{\infty} \frac{\alpha_q^k |u^*|^{\frac{kq}{q-1}}}{k!}$$

can be estimated by means of the following version of the Strauss radial lemma [25]

Lemma 1. *If $u \in L^{p,q}(\mathbb{R}^N)$, $1 < p < \infty$, $1 \leq q < \infty$, then*

$$(21) \quad u^*(r) \leq \left(\frac{q}{p} \right)^{1/q} \frac{\|u\|_{p,q}}{r^{1/p}}$$

Proof. For all $r > 0$ we have

$$\|u\|_{p,q}^q = \int_0^{\infty} |u^*(s)|^{q/p} s^{q/p} \frac{ds}{s} \geq \int_0^r |u^*(s)|^{q/p} s^{q/p} \frac{ds}{s}$$

$$\geq |u^*(r)|^{q/p} \int_0^r s^{q/p} \frac{ds}{s} = |u^*(r)|^{q/p} \frac{p}{q} r^{q/p}$$

and the lemma follows. \square

Thus Lemma 1 yields the following estimate

$$\int_{r_0}^{\infty} |u^*|^{\frac{kq}{q-1}} ds \leq \left(\frac{q}{2} \right)^{\frac{k}{q-1}} \|u\|_{2,q}^{\frac{kq}{q-1}} \int_{r_0}^{\infty} \frac{ds}{s^{\frac{kq}{2(q-1)}}}$$

Now observe that the integral on the right hand side is finite if and only if

$$k \geq 1 + \left[\frac{2(q-1)}{q} \right] = \begin{cases} 1, & \text{if } q < 2 \\ 2, & \text{if } q \geq 2 \end{cases}$$

and we have,

$$\int_{r_0}^{\infty} |u^*|^{\frac{kq}{q-1}} ds \leq r_0 \frac{2(q-1)}{kq - 2(q-1)} \left(\left(\frac{q}{2} \right)^{\frac{1}{q}} \frac{\|u\|_{2,q}}{\sqrt{r_0}} \right)^{\frac{kq}{q-1}}$$

We distinguish three cases according to:

$\boxed{q > 2}$ In this case we have

$$(22) \quad \int_{r_0}^{+\infty} \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 - \alpha_q |u^*|^{\frac{q}{q-1}} \right) ds \\ \leq r_0 (q-1) \sum_{k=2}^{\infty} \frac{\alpha_q^k}{k!} \left(\left(\frac{q}{2} \right)^{\frac{1}{q}} \frac{\|u\|_{2,q}}{\sqrt{r_0}} \right)^{\frac{kq}{q-1}} \leq c(r_0, q)$$

since $\|u\|_{2,q} \leq 1$.

If $\boxed{q < 2}$ it follows that

$$(23) \quad \int_{r_0}^{+\infty} \left(e^{\alpha_q |u^*|^{\frac{q}{q-1}}} - 1 \right) ds \leq r_0 \frac{2(q-1)}{2-q} \sum_{k=1}^{\infty} \frac{\alpha_q^k}{k!} \left(\left(\frac{q}{2} \right)^{\frac{1}{q}} \frac{\|u\|_{2,q}}{\sqrt{r_0}} \right)^{\frac{kq}{q-1}} \leq c(r_0)$$

since $\|u\|_{2,q} \leq 1$.

In the case $\boxed{q = 2}$ we find as in [24],

$$(24) \quad \int_{r_0}^{+\infty} \left(e^{4\pi |u^*|^2} - 1 \right) ds \leq 4\pi \|u\|_2^2 + r_0 \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{4\pi \|u\|_2}{\sqrt{r_0}} \right)^{2k} \leq c(r_0)$$

since $\|u\|_2 = \|u\|_{2,2} \leq 1$.

To estimate the first integral term in (18)–(19), we construct, in the spirit of [6], a function $v(x) \in W_0^1 L^{2,q}(B_{r_1})$, where $B_{r_1} \subset \mathbb{R}^2$ is the ball centered in the origin with radius $r_1 = \sqrt{r_0/\pi}$, such that $u^*(r) - u^*(r_0) \leq v^*(r)$ and $|\nabla v|$ is dominated by $|\nabla u|$ in the sense of Definition 3.

Let $U(x)$ be the function built from $|\nabla u|$ on the level sets of u , i.e. as in (13),

$$(25) \quad \int_{|u|>t} |\nabla u| dx = \int_0^{|\{ |u|>t \}|} U(s) ds$$

Lemma 2. *The following estimate holds:*

$$(26) \quad u^*(r) - u^*(r_0) \leq \frac{1}{\sqrt{4\pi}} \int_r^{r_0} \frac{U(s)}{\sqrt{s}} ds$$

Proof. The proof of this lemma can be found in [14] and we just sketch it. By differentiating (25) we get

$$-\frac{d}{dt} \int_{|u|>t} |\nabla u| dx = -\mu'_u(t) U(\mu_u(t))$$

The Fleming-Rishel formula together with the isoperimetric inequality yield

$$\sqrt{4\pi} \sqrt{\mu_u(t)} \leq \int_{\partial\{|u|>t\}} dH_1(t) = -\frac{d}{dt} \int_{|u|>t} |\nabla u| dx = -\mu'_u(t) U(\mu_u(t)),$$

and in turn

$$-(u^*)'(s) \leq \frac{1}{\sqrt{4\pi}} \frac{U(s)}{\sqrt{s}}$$

from which (26) immediately follows. \square

As we are going to show, Lemma 2 allows us to estimate $u^*(r) - u^*(r_0)$ in terms of the radially decreasing function

$$(27) \quad v(x) := \frac{1}{\sqrt{4\pi}} \int_{\pi|x|^2}^{r_0} \frac{U(t)}{\sqrt{t}} dt$$

Note that in general, $|\nabla v|^* \neq |\nabla u|^*$ but $|\nabla v|$ is dominated by $|\nabla u|$. This fact, together with the next lemma, will enable us to estimate $u(x)$ with a function involving just $|\nabla u|^*$.

Lemma 3. *Let v as in (27) and define*

$$\bar{v}^*(t) := \frac{1}{t} \int_0^t v^*(s) ds$$

Then we have the following estimate

$$(28) \quad \bar{v}^*(t) \leq \frac{1}{\sqrt{4\pi}} \left\{ \int_t^{r_0} |\nabla u|^*(s) \frac{ds}{\sqrt{s}} + \frac{1}{\sqrt{t}} \int_0^t |\nabla u|^*(s) ds \right\}$$

Proof. The proof of this lemma can be found in [6] so that we briefly sketch it. From the definition of $v(x)$ we have,

$$\bar{v}^*(t) = \frac{1}{\sqrt{4\pi}} \left\{ \int_t^{r_0} U(s) \frac{ds}{\sqrt{s}} + \frac{1}{t} \int_0^t U(s) \sqrt{s} ds \right\} \leq \frac{1}{\sqrt{4\pi}} \int_0^{r_0} U(s) g(s, t) ds$$

where

$$g(s, t) = \begin{cases} 1/\sqrt{t}, & 0 \leq s \leq t \\ 1/\sqrt{s}, & t < s \leq r_0 \end{cases}$$

Since $U \prec |\nabla u|$, Proposition 1 implies (28) directly. \square

Since v^* is decreasing we have $v^* \leq \bar{v}^*$; thus, combining (26) with (28) we get

$$(29) \quad u^*(r) - u^*(r_0) \leq \frac{1}{\sqrt{4\pi}} \left\{ \int_r^{r_0} |\nabla u|^*(s) \frac{ds}{\sqrt{s}} + \frac{1}{\sqrt{r}} \int_0^r |\nabla u|^*(s) ds \right\}$$

We set

$$(30) \quad w(r) := \begin{cases} u^*(r) - u^*(r_0), & 0 \leq r \leq r_0 \\ 0, & r > r_0 \end{cases}$$

Next we use the following elementary inequality: for $p > 1$ there exists a constant $\beta_p > 0$ such that

$$(31) \quad (a + b)^p \leq a^p + \beta_p a^{p-1} b + 2^p b^p, \quad \forall a, b \geq 0$$

Combining (30) and (31) and using Lemma 1, we obtain

$$\begin{aligned} (u^*(r))^{\frac{q}{q-1}} &= (w(r) + u^*(r_0))^{\frac{q}{q-1}} \\ &\leq w^{\frac{q}{q-1}}(r) + \beta_q w^{\frac{1}{q-1}} u^*(r_0) + (2u^*(r_0))^{\frac{q}{q-1}} \\ &\leq w^{\frac{q}{q-1}}(r) + \frac{\beta_q^q}{q} w^{\frac{q}{q-1}} (u^*(r_0))^q + \frac{q-1}{q} + (2u^*(r_0))^{\frac{q}{q-1}} \\ &\leq w^{\frac{q}{q-1}}(r) \left(1 + \frac{\beta_q^q}{2} \frac{\|u\|_{2,q}^q}{r_0^{q/2}} \right) + \frac{q-1}{q} + \left(2^{\frac{q-1}{q}} q^{\frac{1}{q}} \frac{\|u\|_{2,q}}{\sqrt{r_0}} \right)^{\frac{q}{q-1}} \\ &=: w^{\frac{q}{q-1}}(r) \left(1 + \frac{\beta_q^q}{2} \frac{\|u\|_{2,q}^q}{r_0^{q/2}} \right) + d(r_0) \end{aligned}$$

hence

$$u^*(r) \leq w(r) \left[1 + \frac{\beta_q^q}{2} \frac{\|u\|_{2,q}^q}{r_0^{q/2}} \right]^{\frac{q-1}{q}} + (d(r_0))^{\frac{q-1}{q}} =: z(r) + (d(r_0))^{\frac{q-1}{q}}$$

and the following estimate holds

$$(32) \quad \int_0^{r_0} \left(e^{\alpha_q (u^*)^{\frac{q}{q-1}}} - 1 \right) dr \leq e^{\alpha_q d(r_0)} \int_0^{r_0} e^{\alpha_q (z(r))^{\frac{q}{q-1}}} dr \\ = r_0 e^{\alpha_q d(r_0)} \int_0^{+\infty} e^{\alpha_q (z(r_0 e^{-t}))^{\frac{q}{q-1}} - t} dt$$

Now note that by (29) we have

$$\begin{aligned} z(r) &= w(r) \left[1 + \frac{\beta_q^q}{2} \frac{\|u\|_{2,q}^q}{r_0^{q/2}} \right]^{\frac{q-1}{q}} \\ &\leq \frac{1}{\sqrt{4\pi}} \left[1 + \frac{\beta_q^q}{2} \frac{\|u\|_{2,q}^q}{r_0^{q/2}} \right]^{\frac{q-1}{q}} \left\{ \int_r^{r_0} |\nabla u|^*(s) \frac{ds}{\sqrt{s}} + \frac{1}{\sqrt{r}} \int_0^r |\nabla u|^*(s) ds \right\} \end{aligned}$$

from which we get

$$\begin{aligned} z(r_0 e^{-t}) &\leq \frac{\sqrt{r_0}}{\sqrt{4\pi}} \left[1 + \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right]^{\frac{q-1}{q}} \left\{ \int_0^t |\nabla u|^*(r_0 e^{-r}) e^{-r/2} dr \right. \\ &\quad \left. + e^{t/2} \int_t^\infty |\nabla u|^*(r_0 e^{-r}) e^{-r} dr \right\} \\ &= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} \phi(r) a(r, t) dr \end{aligned}$$

where

$$(33) \quad a(r, t) := \begin{cases} 0, & \text{if } r \leq 0 \\ e^{(t-r)/2}, & \text{if } t < r < +\infty \\ 1, & \text{if } 0 < r < t \end{cases}$$

and

$$(34) \quad \phi(r) := \begin{cases} \sqrt{r_0} \left[1 + \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right]^{\frac{q-1}{q}} |\nabla u|^*(r_0 e^{-r}) e^{-r/2}, & \text{if } r \geq 0 \\ 0, & \text{if } r < 0 \end{cases}$$

A key ingredient in what follows, is the following Lemma due to D.R. Adams [2, Lemma 1, p. 388].

Lemma 4. *Let $a(s, t)$ be a non-negative measurable function on $\mathbb{R} \times [0, +\infty)$ such that, for some $q \in (1, \infty)$*

$$(35) \quad a(s, t) \leq 1 \quad \text{for a.e. } 0 < s < t$$

$$(36) \quad \sup_{t>0} \left(\int_{-\infty}^0 + \int_t^{+\infty} (a(s, t))^{\frac{q}{q-1}} ds \right)^{(q-1)/q} = \gamma < \infty$$

Then, there exists a constant $C = C(q, \gamma)$ such that for $\phi \geq 0$ satisfying

$$(37) \quad \int_{-\infty}^{+\infty} (\phi(s))^q ds \leq 1$$

then

$$(38) \quad \int_0^{+\infty} e^{-\Psi(t)} dt \leq C$$

where

$$(39) \quad \Psi(t) = t - \left\{ \int_{-\infty}^{+\infty} a(s, t) \phi(s) ds \right\}^{\frac{q}{q-1}}$$

Clearly, $a(r, t)$ defined in (33), enjoys (35). Moreover, for any $1 < q < \infty$ one has

$$\left(\int_{-\infty}^0 + \int_t^{+\infty} \right) (a(r, t))^{\frac{q}{q-1}} dr = \int_t^{+\infty} e^{\frac{q(t-r)}{2(q-1)}} dr = e^{\frac{qt}{2(q-1)}} \int_t^{+\infty} e^{-\frac{qr}{2(q-1)}} dr = \frac{2(q-1)}{q}$$

and (36) holds as well.

Finally, notice that for ϕ as in (34) we have

$$\begin{aligned} \int_{-\infty}^{+\infty} (\phi(r))^q dr &= r_0^{\frac{q}{2}} \left(1 + \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right)^{q-1} \int_0^{+\infty} (|\nabla u|^*(r_0 e^{-r}))^q e^{-rq/2} dr \\ &= \left(1 + \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right)^{q-1} \int_0^{r_0} (|\nabla u|^*(s) s^{1/2})^q \frac{ds}{s} \\ &\leq \left(1 + \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right)^{q-1} \|\nabla u\|_{2,q}^q \\ &\leq \left(1 + \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right)^{q-1} (1 - \|u\|_{2,q}^q) \end{aligned}$$

Taking r_0 sufficiently large,

$$\begin{aligned} \int_{-\infty}^{+\infty} (\phi(r))^q dr &\leq \left(1 + q \frac{\beta_q^q \|u\|_{2,q}^q}{2 r_0^{q/2}} \right) (1 - \|u\|_{2,q}^q) \\ &= 1 - \|u\|_{2,q}^q \left(1 - \frac{q\beta_q^q}{2r_0^{q/2}} \right) - \frac{q\beta_q^q}{2r_0^{q/2}} \|u\|_{2,q}^{2q} \leq 1 \end{aligned}$$

provided that $r_0 \geq \left(\frac{q\beta_q^q}{2} \right)^{\frac{2}{q}}$.

Therefore, Lemma 4 applies to our situation and from (32) we conclude, for any $1 < q < \infty$, that

$$\int_0^{r_0} \left(e^{\alpha_q(u^*)^{\frac{q}{q-1}}} - 1 \right) dr \leq C(q)$$

The above inequality, together with (22), (23) and (24), yields (14)–(15) and the proof of Proposition 2 is now complete.

4. MOSER-TYPE CONCENTRATING SEQUENCES

Our next goal, is to show that inequalities (14)–(15) are sharp, in the sense that if the exponent α_q is replaced by any number $\alpha > \alpha_q$, then the corresponding suprema in (14)–(15) become infinity. We achieve this by constructing, as in [6, 24], an optimal normalized concentrating sequence à la Moser: as we are going to see, optimality in our situation, requires a finer balance in energy estimates than in [6, 24]. The main result of this section is the following

Proposition 3. *Suppose that $\alpha > \alpha_q$. Then, for any domain $\Omega \subseteq \mathbb{R}^2$ and for any $1 < q < \infty$*

$$(40) \quad \sup_{\|u\|_{1,(2,q)} \leq 1} \int_{\Omega} \left(e^{\alpha|u|^{\frac{q}{q-1}}} - 1 - \alpha|u|^{\frac{q}{q-1}} \right) dx = +\infty, \quad \text{if } 2 < q < \infty$$

$$(41) \quad \sup_{\|u\|_{1,(2,q)} \leq 1} \int_{\Omega} \left(e^{\alpha|u|^{\frac{q}{q-1}}} - 1 \right) dx = +\infty, \quad \text{if } 1 < q \leq 2$$

4.1. Proof of Proposition 3. We may assume that $0 \in \Omega$ and that for some $\rho > 0$, $B_{\rho}(0) \subset \Omega$. We construct a modified “Moser sequence” (see [22]), $\{m_n\} \subset W_0^1 L^{2,q}(B_{\rho}(0))$ and then extend by zero on $B_{\rho}^c(0)$, obtaining a sequence in $W^1 L^{2,q}(\mathbb{R}^N)$ which is normalized in the Lorentz-Sobolev norm, i.e. $\|m_n\|_{1,(2,q)} \leq 1$.

Definition 4. *Let us define*

$$(42) \quad m_n(x) = \begin{cases} \frac{n^{\frac{q-1}{q}}}{\sqrt{4\pi}} (1 - \delta_n)^{\frac{q-1}{q}}, & |x|^2 \leq \rho^2 e^{-n} \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} \sqrt{4\pi}} \log \left(\frac{\rho^2}{|x|^2} \right), & \rho^2 e^{-n} < |x|^2 \leq \rho^2 \end{cases}$$

where $\delta_n \rightarrow 0$, as $n \rightarrow \infty$.

From (42) one easily gets the following:

$$(43) \quad m_n^*(s) = \begin{cases} \frac{n^{\frac{q-1}{q}}}{\sqrt{4\pi}} (1 - \delta_n)^{\frac{q-1}{q}}, & s \leq \pi\rho^2 e^{-n} \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} \sqrt{4\pi}} \log\left(\frac{\pi\rho^2}{s}\right), & \pi\rho^2 e^{-n} < s \leq \pi\rho^2 \end{cases}$$

$$(44) \quad |\nabla m_n|(x) = \begin{cases} 0, & |x|^2 \leq \rho^2 e^{-n} \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} \sqrt{\pi} |x|}, & \rho^2 e^{-n} < |x|^2 \leq \rho^2 \end{cases}$$

and then

$$(45) \quad |\nabla m_n|^*(s) = \begin{cases} 0, & \pi\rho^2 (1 - e^{-n}) \leq s \leq \pi\rho^2 \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} \sqrt{s + \pi\rho^2 e^{-n}}}, & 0 \leq s < \pi\rho^2 (1 - e^{-n}) \end{cases}$$

Lemma 5. *As $n \rightarrow \infty$ the following asymptotic estimates hold:*

$$(46) \quad \|\nabla m_n\|_{2,q}^q = (1 - \delta_n)^{q-1} + O\left(\frac{1}{n}\right)$$

$$(47) \quad \|m_n\|_{2,q}^q = (1 - \delta_n)^{q-1} O\left(\frac{1}{n}\right)$$

$$(48) \quad \|m_n\|_{1,(2,q)}^q = 1 + o(\delta_n)$$

where $\delta_n \rightarrow 0$, as $n \rightarrow \infty$.

Proof. Performing the change of variable $t = \pi\rho^2 + se^n$, we have

$$\begin{aligned}
\|\nabla m_n\|_{2,q}^q &= \frac{(1-\delta_n)^{q-1}}{n} \int_0^{\pi\rho^2(1-e^{-n})} \left(\frac{s}{s+\pi\rho^2e^{-n}}\right)^{q/2} \frac{ds}{s} \\
&= \frac{(1-\delta_n)^{q-1}}{n} \int_{\pi\rho^2}^{\pi\rho^2e^n} \left(1-\frac{\pi\rho^2}{t}\right)^{q/2} \frac{dt}{t-\pi\rho^2} \\
(49) \quad &= \frac{(1-\delta_n)^{q-1}}{n} \int_1^{e^n} \left(1-\frac{1}{y}\right)^{q/2} \frac{dy}{y-1} \\
&= (1-\delta_n)^{q-1} \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \rightarrow \infty
\end{aligned}$$

since, as $n \rightarrow \infty$, one has

$$\begin{aligned}
\frac{1}{n} \int_1^{e^n} \left(1-\frac{1}{y}\right)^{q/2} \frac{dy}{y-1} &= \frac{1}{n} \left\{ \int_1^K + \int_K^{e^n} \left(1-\frac{1}{y}\right)^{q/2} \frac{dy}{y-1} \right\} \\
&= O\left(\frac{1}{n}\right) + \frac{1}{n} \int_K^{e^n} \left(1-\frac{1}{y}\right)^{q/2} \frac{dy}{y-1}
\end{aligned}$$

and

$$\frac{1}{n} \int_K^{e^n} \frac{c_{1,q}}{y(y-1)} dy \leq \frac{1}{n} \int_K^{e^n} \left[\left(1-\frac{1}{y}\right)^{q/2} - 1 \right] \frac{dy}{y-1} \leq \frac{1}{n} \int_K^{e^n} \frac{c_{2,q}}{y(y-1)} dy$$

so that

$$\frac{1}{n} \int_K^{e^n} \left(1-\frac{1}{y}\right)^{q/2} \frac{dy}{y-1} = \frac{1}{n} \int_K^{e^n} \frac{dy}{y-1} + O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n}\right)$$

and thus (46) follows. Next we evaluate $\|m_n\|_{2,q}$:

$$\begin{aligned}
\|m_n\|_{2,q}^q &= \frac{n^{q-1}(1-\delta_n)^{q-1}}{(4\pi)^{q/2}} \int_0^{\pi\rho^2e^{-n}} s^{\frac{q}{2}-1} ds \\
&\quad + \frac{(1-\delta_n)^{q-1}}{n(4\pi)^{q/2}} \int_{\pi\rho^2e^{-n}}^{\pi\rho^2} \log^q\left(\frac{\pi\rho^2}{s}\right) s^{q/2} \frac{ds}{s} \\
(50) \quad &= (1-\delta_n)^{q-1} \left\{ \frac{\rho^q}{2^{q-1}} n^{q-1} e^{-\frac{q}{2}n} + \frac{\rho^q}{n2^q} \int_0^n t^q e^{-\frac{q}{2}t} dt \right\} \\
&= (1-\delta_n)^{q-1} O\left(\frac{1}{n}\right)
\end{aligned}$$

and hence (47). Joining (46) and (47) we obtain

$$\begin{aligned} \|m_n\|_{1,(2,q)}^q &= (1 - \delta_n)^{q-1} \left\{ \frac{1}{n} \int_1^{e^n} \left(1 - \frac{1}{y}\right)^{q/2} \frac{dy}{y-1} + \frac{\rho^q}{2^{q-1}} n^{q-1} e^{-\frac{q}{2}n} \right. \\ &\quad \left. + \frac{\rho^q}{n2^q} \int_0^n t^q e^{-\frac{q}{2}t} dt \right\} \\ &= (1 - \delta_n)^{q-1} \left\{ 1 + \frac{1}{n} \int_1^{e^n} \left(\frac{(y-1)^{\frac{q}{2}-1}}{y^{q/2}} - \frac{1}{y} \right) dy + \frac{\rho^q}{2^{q-1}} n^{q-1} e^{-\frac{q}{2}n} \right. \\ &\quad \left. + \frac{\rho^q}{n2^q} \int_0^n t^q e^{-\frac{q}{2}t} dt \right\} \end{aligned}$$

Now we choose δ_n as follows:

$$(51) \quad \delta_n = \frac{1}{n(q-1)} \left\{ \int_1^{e^n} \left(\frac{(y-1)^{\frac{q}{2}-1}}{y^{q/2}} - \frac{1}{y} \right) dy + \frac{\rho^q}{2^q} \int_0^n t^q e^{-\frac{q}{2}t} dt \right\}$$

so that, as $n \rightarrow \infty$,

$$\|m_n\|_{1,(2,q)}^q = (1 - \delta_n)^{q-1} (1 + (q-1)\delta_n + o(\delta_n)) = 1 + o(\delta_n)$$

that is (48). □

Notice that, by the calculations carried out in (49) and (50),

$$(52) \quad \delta_n = O\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty$$

and thus we have

$$(53) \quad \|m_n\|_{1,(2,q)}^q = 1 + o(1/n), \quad \text{as } n \rightarrow \infty$$

Set

$$u_n(x) := \frac{m_n(x)}{\|m_n\|_{1,(2,q)}}$$

Clearly, $\|u_n\|_{1,(2,q)} = 1$ and moreover, by (52) and (53), we have

$$\begin{aligned} \int_{B_\rho(0)} e^{\alpha|u_n|^{\frac{q}{q-1}}} dx &\geq \exp\left(\frac{\alpha}{\alpha_q} \frac{n(1-\delta_n)}{\|m_n\|_{S,2,q}^{\frac{q}{q-1}}} - n + \log(\pi\rho^2)\right) \\ &= \exp\left(\frac{\alpha}{\alpha_q} n(1-\delta_n)(1+o(\delta_n)) - n + \log(\pi\rho^2)\right) \\ &= \exp\left(\left(\frac{\alpha}{\alpha_q} - 1\right)n + O(1)\right) \rightarrow +\infty, \quad n \rightarrow \infty \end{aligned}$$

if $\alpha > \alpha_q$. This inequality implies directly (41).

For $q > 2$, notice that

$$\alpha \int_{B_\rho(0)} |u_n|^{\frac{q}{q-1}} dx = \frac{\alpha}{\alpha_q} \frac{\pi\rho^2(1-\delta_n)}{\|m_n\|_{1,(2,q)}^{\frac{q}{q-1}}} \left\{ ne^{-n} + \frac{1}{n^{q-1}} \int_0^n t^{\frac{q}{q-1}} e^{-t} dt \right\} \rightarrow 0$$

as $n \rightarrow \infty$, which yields (40) and the proof of Proposition 3 is now complete.

5. THE EXTENSION TO DIMENSION $N \geq 3$

In this section, relying on the ideas previously introduced, we prove Theorem 2 which extends Theorem 1 to higher dimensions. We consider just the case $q \neq N$, since the case $q = N$ has been already covered in [19]. In the proof of Theorem 2 we essentially follow the line used in proving Theorem 1, therefore we are going to give just some highlights taking care of stressing the differences between. In what follows we set for simplicity $\alpha_q := \alpha_{N,q}$.

5.1. Proof of Theorem 2. As in Theorem 1, it is enough to show that

$$(54) \quad \sup_{\|u\|_{1,(N,q)} \leq 1} \int_{\mathbb{R}^N} \Phi\left(\alpha_q |u|^{\frac{q}{q-1}}\right) dx \leq C$$

holds for $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Applying symmetrization, we estimate

$$\begin{aligned} \int_{\mathbb{R}^N} \Phi\left(\alpha_q |u|^{\frac{q}{q-1}}\right) dx &= \int_0^\infty \Phi\left(\alpha_q |u^*|^{\frac{q}{q-1}}\right) ds \\ &= \left(\int_0^{r_0} + \int_{r_0}^{+\infty} \right) \Phi\left(\alpha_q |u^*|^{\frac{q}{q-1}}\right) ds = I_1 + I_2 \end{aligned}$$

We write the second integral as

$$(55) \quad I_2 = \sum_{k=k_0+1}^{\infty} \int_{r_0}^{\infty} \frac{\alpha_q^k |u^*|^{\frac{kq}{q-1}}}{k!} ds$$

Each term in (55) can be estimated by Lemma 1, obtaining for $k \geq 1 + k_0$

$$\begin{aligned} \int_{r_0}^{\infty} |u^*|^{\frac{kq}{q-1}} ds &\leq \left(\frac{q}{N}\right)^{\frac{k}{q-1}} \|u\|_{N,q}^{\frac{kq}{q-1}} \int_{r_0}^{\infty} \frac{ds}{s^{\frac{kq}{N(q-1)}}} \\ &\leq r_0 \frac{N(q-1)}{kq - N(q-1)} \left(\left(\frac{q}{N}\right)^{\frac{1}{q}} \frac{\|u\|_{N,q}}{r_0^{1/N}} \right)^{\frac{kq}{q-1}} \end{aligned}$$

Hence

$$(56) \quad \int_{r_0}^{+\infty} \Phi \left(\alpha_q |u^*|^{\frac{q}{q-1}} \right) ds \leq r_0(q-1) \sum_{k=k_0+1}^{\infty} \frac{\alpha_q^k}{k!} \left(\left(\frac{q}{N}\right)^{\frac{1}{q}} \frac{\|u\|_{N,q}}{r_0^{1/N}} \right)^{\frac{kq}{q-1}} \leq c(r_0)$$

since $\|u\|_{N,q} \leq 1$.

In order to estimate the first integral I_1 , we construct a function $v(x) \in W_0^1 L^{N,q}(B_{r_1})$, where $B_{r_1} \subset \mathbb{R}^N$ is the ball with center in the origin and radius $r_1 = \sqrt{r_0/\pi}$, such that

$$(57) \quad u^*(r) - u^*(r_0) \leq v^*(r)$$

and such that $|\nabla v|$ is dominated by $|\nabla u|$. Indeed, a closer inspection of the proof of Lemma 2, shows that

$$(58) \quad u^*(r) - u^*(r_0) \leq \frac{1}{N\omega_N^{1/N}} \int_r^{r_0} U(s) s^{1/N} \frac{ds}{s}$$

with $U(x)$ as in (25): thus (57) holds by choosing the radially decreasing function

$$(59) \quad v(x) := \frac{1}{N\omega_N^{1/N}} \int_{\omega_N|x|^N}^{r_0} U(s) s^{1/N} \frac{ds}{s}$$

As in Lemma 3 one has

$$(60) \quad u^*(r) - u^*(r_0) \leq \frac{1}{N\omega_N^{1/N}} \left\{ \int_r^{r_0} |\nabla u|^*(s) s^{1/N} \frac{ds}{s} + \frac{1}{r^{1-1/N}} \int_0^r |\nabla u|^*(s) ds \right\}$$

and if $w(r)$ is defined as in (30), by performing the same calculations as in Section 3.1, we get the following estimates:

$$u^*(r) \leq w(r) \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{q/N}} \right]^{\frac{q-1}{q}} + d(r_0)^{\frac{q-1}{q}} =: z(r) + d(r_0)^{\frac{q-1}{q}}$$

and

$$(61) \quad \int_0^{r_0} \left(e^{\alpha_q (u^*)^{\frac{q}{q-1}}} - 1 \right) dr \leq e^{\alpha_q d(r_0)} \int_0^{+\infty} e^{\alpha_q z^{\frac{q}{q-1}} (r_0 e^{-t}) - t} dt$$

On the other hand, by (60),

$$\begin{aligned} z(r) &= w(r) \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{\frac{q-1}{q}} \\ &\leq \frac{1}{N \omega_N^{\frac{1}{N}}} \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{\frac{q-1}{q}} \left\{ \int_r^{r_0} |\nabla u|^*(s) s^{\frac{1}{N}} \frac{ds}{s} + \frac{1}{r^{1-N}} \int_0^r |\nabla u|^*(s) ds \right\} \end{aligned}$$

so that

$$\begin{aligned} z(r_0 e^{-t}) &\leq \frac{r_0^{1/N}}{N \omega_N^{\frac{1}{N}}} \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{\frac{q-1}{q}} \left\{ \int_0^t |\nabla u|^*(r_0 e^{-r}) e^{-\frac{r}{N}} dr \right. \\ &\quad \left. + e^{t \frac{N-1}{N}} \int_t^{+\infty} |\nabla u|^*(r_0 e^{-r}) e^{-r} dr \right\} \\ &= \frac{1}{N \omega_N^{\frac{1}{N}}} \int_{-\infty}^{+\infty} \phi(r) a(r, t) dr \end{aligned}$$

where we have set

$$(62) \quad a(r, t) := \begin{cases} 0, & \text{if } r \leq 0 \\ e^{(t-r)(N-1)/N}, & \text{if } t < r < +\infty \\ 1, & \text{if } 0 < r < t \end{cases}$$

and

$$(63) \quad \phi(r) := \begin{cases} r_0^{\frac{1}{N}} \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{\frac{q-1}{q}} |\nabla u|^*(r_0 e^{-r}) e^{-\frac{r}{N}}, & \text{if } r \geq 0 \\ 0, & \text{if } r < 0 \end{cases}$$

The next step consists of applying Adams' Lemma 4. Clearly, the function $a(r, t)$ in (62) satisfies (35). Moreover, for $1 < q < \infty$

$$\left(\int_{-\infty}^0 + \int_t^{+\infty} \right) (a(r, t))^{\frac{q}{q-1}} dr = \int_t^{+\infty} e^{\frac{q(t-r)}{N(q-1)}} dr = \frac{N(q-1)}{q}$$

and also (36) holds.

Now observe that

$$\begin{aligned}
\int_{-\infty}^{+\infty} (\phi(r))^q dr &= r_0^{\frac{q}{N}} \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{q-1} \int_0^{+\infty} (|\nabla u|^*(r_0 e^{-r}))^q e^{-\frac{rq}{N}} dr \\
&= \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{q-1} \|\nabla u\|_{N,q}^q \\
&\leq \left[1 + \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right]^{q-1} \left[1 - \|u\|_{N,q}^q \right]
\end{aligned}$$

By taking r_0 large enough, we get

$$\int_{-\infty}^{+\infty} (\phi(r))^q dr \leq \left[1 + q \frac{\beta_q^q \|u\|_{N,q}^q}{N r_0^{\frac{q}{N}}} \right] \left[1 - \|u\|_{N,q}^q \right] \leq 1$$

and inequality (12) follows from Adams' Lemma 4 and (61).

It remains to prove the sharpness of (12) for which we use a suitable modification of the Moser-type sequence constructed in Section 4.

Let us define

$$(64) \quad m_n(x) := \begin{cases} \frac{n^{\frac{q-1}{q}} (1 - \delta_n)^{\frac{q-1}{q}}}{N \omega_N^{\frac{1}{N}}}, & |x|^N \leq \rho^N e^{-n} \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} N \omega_N^{\frac{1}{N}}} \log \left(\frac{\rho^N}{|x|^N} \right), & \rho^N e^{-n} < |x|^N \leq \rho^N \end{cases}$$

where $\delta_n \rightarrow 0$, as $n \rightarrow \infty$, will be determined later on. We have

$$(65) \quad m_n^*(s) = \begin{cases} \frac{n^{\frac{q-1}{q}} (1 - \delta_n)^{\frac{q-1}{q}}}{N \omega_N^{\frac{1}{N}}}, & s \leq \omega_N \rho^N e^{-n} \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} N \omega_N^{\frac{1}{N}}} \log \left(\frac{\omega_N \rho^N}{s} \right), & \omega_N \rho^N e^{-n} < s \leq \omega_N \rho^N \end{cases}$$

and

$$(66) \quad |\nabla m_n|^*(s) = \begin{cases} 0, & \omega_N \rho^N (1 - e^{-n}) \leq s \leq \omega_N \rho^N \\ \frac{(1 - \delta_n)^{\frac{q-1}{q}}}{n^{\frac{1}{q}} \sqrt{s + \omega_N \rho^N e^{-n}}}, & 0 \leq s < \omega_N \rho^N (1 - e^{-n}) \end{cases}$$

By performing the change of variable $t = \omega_N \rho^N + s e^n$, we estimate

$$(67) \quad \begin{aligned} \|\nabla m_n\|_{N,q}^q &= \frac{(1 - \delta_n)^{q-1}}{n} \int_{\omega_N \rho^N}^{\omega_N \rho^N e^n} \left(1 - \frac{\omega_N \rho^N}{t}\right)^{\frac{q}{N}} \frac{dt}{t - \omega_N \rho^N} \\ &= \frac{(1 - \delta_n)^{q-1}}{n} \int_1^{e^n} \left(1 - \frac{1}{y}\right)^{\frac{q}{N}} \frac{dy}{y-1} \\ &= (1 - \delta_n)^{q-1} \left(1 + O\left(\frac{1}{n}\right)\right) \end{aligned}$$

as $n \rightarrow \infty$, since

$$\frac{1}{n} \int_1^{e^n} \left(1 - \frac{1}{y}\right)^{\frac{q}{N}} \frac{dy}{y-1} = O\left(\frac{1}{n}\right) + \frac{1}{n} \int_K^{e^n} \left(1 - \frac{1}{y}\right)^{\frac{q}{N}} \frac{dy}{y-1}$$

and

$$\frac{1}{n} \int_K^{e^n} \frac{c_{1,q}}{y(y-1)} dy \leq \frac{1}{n} \int_K^{e^n} \left[\left(1 - \frac{1}{y}\right)^{q/N} - 1 \right] \frac{dy}{y-1} \leq \frac{1}{n} \int_K^{e^n} \frac{c_{2,q}}{y(y-1)} dy$$

so that

$$\frac{1}{n} \int_K^{e^n} \left(1 - \frac{1}{y}\right)^{q/N} \frac{dy}{y-1} = \frac{1}{n} \int_K^{e^n} \frac{dy}{y-1} + O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n}\right)$$

Whereas,

$$(68) \quad \begin{aligned} \|m_n\|_{N,q}^q &= \frac{n^{q-1} (1 - \delta_n)^{q-1}}{\left(N \omega_N^{\frac{1}{N}}\right)^q} \int_0^{\omega_N \rho^N e^{-n}} s^{\frac{q}{N}-1} ds \\ &\quad + \frac{(1 - \delta_n)^{q-1}}{n \left(N \omega_N^{\frac{1}{N}}\right)^q} \int_{\omega_N \rho^N e^{-n}}^{\omega_N \rho^N} \left(\log\left(\frac{\omega_N \rho^N}{s}\right)\right) s^{\frac{q}{N}} \frac{ds}{s} \\ &= (1 - \delta_n)^{q-1} \left\{ \frac{\rho^q}{N^{q-1}} n^{q-1} e^{-\frac{q}{N}n} + \frac{\rho^q}{n N^q} \int_0^n t^q e^{-\frac{q}{N}t} dt \right\} \\ &= (1 - \delta_n)^{q-1} O\left(\frac{1}{n}\right), \quad \text{as } n \rightarrow \infty \end{aligned}$$

Joining (67) and (68) we get

$$\begin{aligned} \|m_n\|_{1,(N,q)}^q &= (1 - \delta_n)^{q-1} \left\{ \frac{1}{n} \int_1^{e^n} \left(1 - \frac{1}{y}\right)^{\frac{q}{N}} \frac{dy}{y-1} + \frac{\rho^q}{N^{q-1}} n^{q-1} e^{-\frac{q}{N}n} \right. \\ &\quad \left. + \frac{\rho^q}{nN^q} \int_0^n t^q e^{-\frac{q}{N}t} dt \right\} \\ &= (1 - \delta_n)^{q-1} \left\{ 1 + \frac{1}{n} \int_1^{e^n} \left(\frac{(y-1)^{\frac{q}{N}-1}}{y^{q/N}} - \frac{1}{y} \right) dy \right. \\ &\quad \left. + \frac{\rho^q}{N^{q-1}} n^{q-1} e^{-\frac{q}{N}n} + \frac{\rho^q}{nN^q} \int_0^n t^q e^{-\frac{q}{N}t} dt \right\} \end{aligned}$$

Now we choose δ_n as follows:

$$(69) \quad \delta_n = \frac{1}{n(q-1)} \left\{ \int_1^{e^n} \left(\frac{(y-1)^{\frac{q}{N}-1}}{y^{q/N}} - \frac{1}{y} \right) dy + \frac{\rho^q}{N^q} \int_0^n t^q e^{-\frac{q}{N}t} dt \right\}$$

and notice that

$$(70) \quad \delta_n = O\left(\frac{1}{n}\right), \quad \text{as } n \rightarrow \infty$$

Finally we obtain

$$\|m_n\|_{1,(N,q)}^q = (1 - \delta_n)^{q-1} (1 + (q-1)\delta_n + o(\delta_n)) = 1 + o(\delta_n) = 1 + o(1/n)$$

Let

$$u_n(x) := \frac{m_n(x)}{\|m_n\|_{1,(N,q)}}$$

so that $\|u_n\|_{1,(N,q)} = 1$ whereas (70) yields

$$\begin{aligned} \int_{B_\rho} e^{\alpha|u_n|^{\frac{q}{q-1}}} dx &\geq \exp\left(\frac{\alpha}{\alpha_q} \frac{n(1-\delta_n)}{\|m_n\|_{1,(N,q)}^{\frac{q}{q-1}}} - n + \log(\omega_N \rho^N)\right) \\ &= \exp\left(\frac{\alpha}{\alpha_q} n(1-\delta_n)(1 + o(\delta_n)) - n + \log(\omega_N \rho^N)\right) \\ &= \exp\left(\left(\frac{\alpha}{\alpha_q} - 1\right)n + O(1)\right) \rightarrow +\infty, \quad \text{as } n \rightarrow \infty \end{aligned}$$

if $\alpha > \alpha_q$. We conclude the proof of Theorem 2 by observing that, for k satisfying

$$1 \leq k \leq \left\lfloor \frac{N(q-1)}{q} \right\rfloor \leq N-1$$

the following holds

$$\alpha^k \int_{B_\rho} |u_n|^{\frac{kq}{q-1}} dx = \frac{\alpha^k \omega_N \rho^N (1 - \delta_n)^k}{\alpha_q^k \|m_n\|_{1,(N,q)}^{\frac{kq}{q-1}}} \left\{ n^k e^{-n} + \frac{1}{n^{k(q-1)}} \int_0^n t^{\frac{kq}{q-1}} e^{-t} dt \right\} \rightarrow 0$$

as $n \rightarrow \infty$.

6. FINAL REMARKS

So far, we have considered the exponent q in the range $1 < q < \infty$. Clearly, for $q = 1$ as well as for $q = \infty$, the situation in Theorem 2 degenerates and thus we next consider these borderline situations. Let us recall from [5] that, for $1 \leq q \leq N$, the following inequality holds

$$(71) \quad u^\#(x) \leq \frac{\|\nabla u\|_{N,q}}{N^{\frac{1}{q}} \omega_N^{\frac{1}{N}}} \left(\log \left(\frac{R_0}{|x|} \right) \right)^{\frac{q-1}{q}}, \quad x \in (\text{supp } u)^\#$$

which is valid for $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. In particular, for $q = 1$, we get directly from (71)

$$(72) \quad \|u\|_\infty \leq \frac{\|u\|_{1,(N,1)}}{N \omega_N^{\frac{1}{N}}}$$

In order to handle the case $q = \infty$ we recall from [17] the definition of the so-called *weak- L^p* spaces, precisely the Lorentz spaces $L^{p,\infty}$, defined for $1 < p \leq \infty$ by means of the following norm:

$$\| |u| \|_{p,\infty} := \sup_{t>0} t^{\frac{1}{p}} \bar{u}(t)$$

or similarly to the case $q < \infty$, in terms of the equivalent quasi-norm:

$$(73) \quad \|u\|_{p,\infty} := \sup_{t>0} t^{\frac{1}{p}} u^*(t)$$

(Notice that $L^{\infty,\infty} \equiv L^\infty$ and that in the Lorentz scale (10) one actually has: $\dots \subset L^{p,q_2} \subset L^{p,\infty} \subset \dots$)

Set $\alpha_{N,\infty} := N \omega_N^{\frac{1}{N}}$ and let $\alpha, r_0 > 0$, assume also that

$$\|u\|_{N,\infty} + \|\nabla u\|_{N,\infty} \leq 1$$

then we have by (73)

$$\int_{r_0}^\infty \left(e^{\alpha|u^*|} - 1 \right) ds \leq \sum_{k=1}^\infty \frac{\alpha^k}{k!} \|u\|_{N,\infty}^k \int_{r_0}^\infty \frac{ds}{s^{\frac{k}{N}}}$$

where the integral on right hand side is finite provided that $k > N$. Thus the following bound holds:

$$(74) \quad \int_{r_0}^{\infty} \left(e^{\alpha|u^*|} - 1 - \sum_{k=1}^N \frac{\alpha^k}{k!} |u^*|^k \right) ds \leq c(r_0)$$

since $\|u\|_{N,\infty} \leq 1$.

Next reasoning as in Section 5.1 we get directly from estimate (60):

$$u^*(r) - u^*(r_0) \leq \frac{\|\nabla u\|_{N,\infty}}{\alpha_{N,\infty}} \left(\log \frac{r_0}{r} + \frac{N}{N-1} \right)$$

which yields, by definition (73) and since $\|\nabla u\|_{N,\infty} \leq 1$, the following

$$(75) \quad u^*(r) \leq \frac{1}{r^{\frac{1}{N}}} + \frac{1}{\alpha_{N,\infty}} \left(\log \frac{r_0}{r} + \frac{N}{N-1} \right)$$

and in turn we eventually get

$$(76) \quad \int_0^{r_0} \left(e^{\alpha|u^*|} - 1 \right) ds \leq r_0^{\frac{\alpha}{\alpha_{N,\infty}}} e^{\frac{\alpha N}{\alpha_{N,\infty}(N-1)}} \int_0^{r_0} \frac{e^{\alpha r^{-\frac{1}{N}}}}{r^{\frac{\alpha}{\alpha_{N,\infty}}}} dr \leq c(r_0, \alpha) < \infty$$

provided that $\alpha < \alpha_{N,\infty}$. We have actually proved the following

Corollary 1. *Let $\alpha < \alpha_{N,\infty}$, then the following inequality holds*

$$\sup_{\|u\|_{1,(N,\infty)} \leq 1} \int_{\mathbb{R}^N} \left(e^{\alpha|u|} - 1 - \sum_{k=1}^N \frac{\alpha^k}{k!} |u|^k \right) dx \leq C(\alpha) < \infty$$

provided that $\alpha < \alpha_{N,\infty}$.

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