Nonexistence results for a class of nonlinear elliptic equations involving critical Sobolev exponents

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 3$; consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda g(x,u) + |u|^{2^*-2} u, & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

where $2^* = 2N/(N-2)$ is critical from the viewpoint of the Sobolev embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, and $g(x,u)$ is a lower-order perturbation of u^{2^*-1}, in the sense that $\lim_{u\to+\infty} g(x,u)/u^{2^*-1} = 0$. As well known, if g satisfies suitable assumptions, solutions of (1) correspond to critical points of the functional

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega G(x,u)\, dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\, dx,$$

where $G(x,u) = \int_0^u g(x,t)\, dt$. Since the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ is not compact, the functional Ψ does not satisfy the Palais-Smale condition: hence

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the standard variational arguments do not apply. For equations with critical growth, nontrivial solution may non exists: a well-known nonexistence result due to Pohozaev [6] asserts that if Ω is starshaped and $\lambda \leq 0$ there is no solution (different from the trivial one) of the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

In recent years this situation of lack of compactness has been extensively investigated (see for example [5]); according to the behaviour of g and the kind of results one wants to prove, topological or variational methods turn out to be more appropriate. When g is superlinear, for example $g = |u|^{p-1} u$, $1 < p < 2^* - 1$, variational tools, such as minimax arguments, provide the existence of a nontrivial positive solution; on the contrary when g is sublinear, for example $g = |u|^{p-1} u$, $0 < p < 1$, sub- and super-solutions are quite convenient. In particular we recall the following known existence results for problem (1):

- The first existence result is due to Brezis-Nirenberg [2]; in a pioneering result, they showed that, when $g(x, u) = u$, there exists a nontrivial positive solution if $\lambda \in (\lambda^*, \lambda_1)$, with $\lambda^* = 0$ for $N \geq 4$ and $0 < \lambda^*(\Omega) < \lambda_1$ for $N = 3$ (λ_1 denoting the first eigenvalue of $-\Delta$ relative to the homogeneous Dirichlet problem in Ω). In the same work they also proved an existence result for equation (1) when g, roughly speaking, has a linear or superlinear growth near zero and near infinity: in this case there is again bifurcation from infinity in $\lambda = 0$ for $N \geq 4$, whereas for $N = 3$ it can not be guaranteed in the entire subcritical growth range of the term g.
- Later, Ambrosetti-Brezis-Cerami [1] established the existence of two positive solutions for $0 < \lambda < \Lambda$ when $g = u^q$ with $0 < q < 1$ and $N \geq 3$, thanks to the combined effects of the sublinear and superlinear terms. The first solution is found using sub- and super-solutions; in contrast with the pure concave case, a second positive solution is found by variational arguments. Moreover, they proved that the first solution, u_λ, is such that $\|u_\lambda\|_\infty \to 0$ as $\lambda \downarrow 0$, while the second solution, w_λ, (if Ω is strictly starshaped) has a nonlimited norm, that is, $\|w_\lambda\|_\infty \to \infty$ as $\lambda \downarrow 0$.

One may ask if the superlinear/sublinear growth of the subcritical term can be weakened in these existence results, e. g., considering subcritical terms presenting superlinear or subliner asymptotic behaviour near zero or near infinity. In fact, the proofs presented by Brezis-Nirenberg in [2] and by Ambrosetti-Brezis-Cerami [1] can be generalized with some technicalities to subcritical terms presenting, respectively, a superlinear or sublinear asymptotic behaviour near the origin: for example, it is not hard to prove the existence of a positive solution for problem (1) if $N \geq 5$ and $g(x,u)$ satisfies the following assumptions

$$\begin{cases} (i) \quad g(x,u) = |u|^{p-1} u \qquad \text{for} \quad |u| < 1, \quad x \in \Omega, \quad p > 1 \\ \\ (ii) \quad \exists \delta > 0 : g(x,s) \geq 0 \qquad \forall |x| < \delta, \quad \forall s > 0, \end{cases} \tag{3}$$

and the existence of two positive solutions for $\lambda \in (0, \Lambda)$ if $N \geq 3$ and $g(x,u)$ satisfies

$$\begin{cases} (i) \quad g(x,u) = |u|^{p-1} u \qquad \text{for} \quad |u| < 1, \quad x \in \Omega, \\ \qquad\qquad 0 < p < 1, \quad p < p_1 < 2^* - 1 \\ (ii) \ |u|^{p-1} u \leq g(x,u) \leq |u|^{p_1 - 1} u \quad \text{for} \quad |u| \geq 1, \quad x \in \Omega. \end{cases} \tag{4}$$

We note that the behaviour of the subcritical term near the origin seems to determine the structure of bifurcation from infinity for problem (1): that is, it seems not possible to obtain similar existence results assuming only superlinear or sublinear growth near infinity. In lower dimensions, however, the effect of the pure convex/concave behaviour of the subcritical term assumes an increasing role, that can not be replaced by the analogous asymptotic behaviour of g near zero: for example, in the pure convex case considered in [2] the existence results are valid for $N \geq 4$, whereas for $N = 3$ a nonexistence result is given; assuming convexity near the origin, instead, the existence results can be extended, in general, only for $N \geq 5$, as we will prove exhibiting a counterexample. The aim of this paper is to point out the difference between the pure convex/concave case and the case of convex/concave growth of the subcritical term near the origin in lower dimensions: based on a celebrated Identity due to Pohozaev [6], we construct special classes of nonlinear problems which do not have nontrivial solutions bifurcating from infinity in $\lambda = 0$ (if the domain Ω is strictly starshaped), according to the behaviour of the subcritical term $g(x,u)$ and to the dimension N. In particular, we prove that the first critical dimension is $N = 4$, which is somehow in contrast with the pure convex case considered in [2]. We remark that the class of subcritical terms presented here has superlinear growth near the origin, whereas the growth near infinity can be sublinear or superlinear; sublinear growth near the origin, instead, determines bifurcation from infinity for all $N \geq 3$, either for convex or for concave behaviour of g near infinity, as one can prove following [1] with slight modifications: that is, the role of the asymptotic behaviour of the subcritical term $g(x,u)$ near infinity does not determine the structure of bifurcation of problem (1).

2 Recalls from potential theory and elliptic estimates

Let Ω be a bounded (smooth) domain in $\mathbb{R}^N$, with $N \geq 3$. We will exhibit two classes of subcritical terms $g(x,u)$ such that problem (1) does not admit any positive solution when λ is close to zero and $N = 3, 4$. The proofs rely on the so-called *Pohozaev's identity* [6]: suppose u is a smooth function satisfying

$$\begin{cases} -\Delta u = f(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{5}$$

where g is a continuous function on $\mathbb{R}$ and Ω is a (smooth) starshaped domain. Then we have

$$\left(1 - \frac{1}{2}n\right) \int_{\Omega} f(u) \cdot u dx + n \int_{\Omega} F(u)\, dx = \frac{1}{2} \int_{\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu}\right)^2 ds \tag{6}$$

where

$$F(u) = \int_0^u f(t)\, dt$$

and ν denotes the outward normal to $\partial\Omega$. We will combine the Pohozaev's identity (6) together with some standard elliptic inequalities and the weak interpolation inequality, which we briefly recall in the following (see [4]).
Let us consider a domain $\Omega \subseteq \mathbb{R}^N$; denote with $D(\Omega)$ the space of the test functions and with $D'(\Omega)$ the space of distributions, that is, the dual space of $D(\Omega)$. Let us recall the definition of the Green's functions for the Poisson's equation in $\mathbb{R}^N$,

$$\begin{aligned} G(x) &= -\frac{1}{2\pi} \ln|x| & N = 2 \\ G(x) &= \frac{1}{(N-2)\mu(\mathbb{S}^{N-1})} |x|^{2-N} & N \neq 2 \end{aligned} \tag{7}$$

where $\mu(\mathbb{S}^{N-1})$ is the area of the unit sphere $\mathbb{S}^{N-1} \subseteq \mathbb{R}^N$. It is well known that for every $u \in L^1_{\text{loc}}$, the function

$$k_u(x) = (G * u)(x) = \int_{\Omega} G(x-y) u(y) dy$$

satisfies

$$\begin{aligned} & k_u \in L^1_{\text{loc}}(\Omega) \\ & -\triangle k_u = u \quad \in D'(\Omega) \end{aligned} \tag{8}$$

if the function $y \mapsto G(x-y)u(y)$ is summable over Ω for almost every x. On the other hand, applying the Young's inequality

$$\|g * h\|_p \leq C_{q,r,p,N}\|g\|_q\|h\|_r \quad \text{if} \quad \frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}, \quad p, q, r \geq 1$$

with $g = G, h = u$ and $r = 1$, we have that

$$\|k_u\|_p \leq C_{q,r,p,N}\|G\|_p\|u\|_1. \tag{9}$$

Therefore, combining (7),(8) and (9) we can conclude that the operator $\triangle^{-1}$ is bounded from L^1 to L^p with $p \in [1, 3)$ if $N = 3$, and from L^1 to L^p with $p \in [1, 2)$ if $N = 4$; that is, for every $v \in L^1$ there is $u \in L^p$ (with p satisfying the previous conditions) such that

$$\begin{aligned} & \triangle u = v \\ & \|u\|_p \leq C_{p,N}\|v\|_1 = C_{p,N}\|\triangle u\|_1. \end{aligned} \tag{10}$$

If $p = 3$ and $N = 3$, or, respectively, if $p = 2$ and $N = 4$, (10) are not verified; in this case, however, we can apply the notion of weak L^p spaces (see [4]). Consider the space of all measurable functions u such that

$$[u]_{q,w} = \sup_{\alpha > 0} \ \alpha \cdot \mu\{x : |u(x)| > \alpha\}^{1/q} < \infty; \tag{11}$$

this space is called weak L^q-space $L^q_w(\mathbb{R}^N)$. Any function in $L^q(\mathbb{R}^N)$ is in $L^q_w(\mathbb{R}^N)$: simply note that

$$\|u\|_q^q \geq \int_{|u|>\alpha} |u(x)|^q dx \geq \alpha^q \cdot \mu\{x : |u(x)| > \alpha\}$$

so that

$$\|u\|_q^q \geq [u]^q_{q,w}.$$

The expression (11) does not define a norm; nevertheless, there is an alternative expression, equivalent to (11), that is indeed a norm: it is given by

$$\|u\|_{q,w} = \sup_A \frac{1}{\mu(A)^{1/r}} \int_A |u(x)| dx. \tag{12}$$

where $1/q + 1/r = 1$ and A denotes an arbitrary measurable set of measure $\mu(A) < \infty$. In particular, $u(x) = |x|^{-\lambda}$ is in $L^q_w(\mathbb{R}^N)$ with $q = N/\lambda$, $N > \lambda > 0$ and

$$\|u\|_{N/\lambda,w} = \frac{N}{N-\lambda} \left[\frac{\mu(\mathbb{S}^{N-1})}{N}\right]^{\lambda/N}.$$

The weak Young inequality states that for $g \in L^q_w(\mathbb{R}^N)$ and $\infty > p, q, r > 1$ with $1/p + 1/q + 1/r = 2$, the following inequality holds:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)g(x-y)h(y)dxdy \leq C_{p,q,r}\|f\|_p\|g\|_{q,w}\|h\|_r; \tag{13}$$

taking $\lambda = N/q$ and $g(x) = |x|^{-\lambda}$ the weak Young inequality (13) is equivalent to the Hardy-Littlewod-Sobolev inequality,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)|x-y|^{-\lambda}h(y)dxdy \leq C_{N,\lambda,p}\|f\|_p\|h\|_r$$

with $p, r > 1, 0 < \lambda < N$ and $1/p + \lambda/N + 1/r = 2$. In particular, the sharp constant in the weak Young inequality is the same as for the Hardy-Littlewood-Sobolev inequality. Observe that we can also view the Young inequality as the statement that the convolution is a bounded map from $L^p(\mathbb{R}^N) \times L^q_w(\mathbb{R}^N)$ to $L^s(\mathbb{R}^N)$, that is

$$\|f * g\|_s \leq C_{q,s,p,N}\|g\|_{q,w}\|f\|_p \quad \text{if} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}, \quad p, q, s > 1. \tag{14}$$

A final inequality involving the L^q_w spaces is the weak interpolation inequality: if $u \in L^p_w \cap L^r$, then

$$\|u\|_q \leq K_{q,r,p,N}\|u\|^a_{p,w}\|u\|^{1-a}_r \quad \text{with} \quad \frac{1}{q} = \frac{a}{p} + \frac{1-a}{r}. \tag{15}$$

This inequality will allow us to combine the estimates obtained from the Pohozaev's identity with the elliptic estimates (10).

3 Nonexistence results

In this section we construct two classes of nonlinear elliptic problems with critical growth which don't admit any positive solution near $\lambda = 0$. The proof of nonexistence is based on Pohozaev's identity and on the elliptic estimates presented in the previous section. From now on suppose Ω is strictly starshaped about the origin, so that $(x \cdot \nu) > c > 0$ a.e. on $\partial\Omega$. We discuss separately the two cases, $N = 3$ and $N = 4$.

3.1 The case $N = 3$.

We assume here that $N = 3$ and

$$g(u) = \begin{cases} |u|^{p-1} \cdot u & |u| < 1, \ \ 1 < p \\ |u|^{q-1} \cdot u & |u| \geq 1, \ \ 0 < q \leq 3 \end{cases} \tag{16}$$

Then we have the following result.

Theorem 1 *Let Ω be strictly starshaped about the origin; suppose that u is a solution of problem (1), with g given by (16). Then*

$$\lambda \geq \lambda_0 \left(q, p, \Omega\right) > 0$$

if $1 < p,\ 0 < q \leq 3$.

Proof of Theorem 1. This Theorem is a slight extension of Theorem 2.4 in [2], so we will be brief. By Pohozaev's identity (6), since Ω is strictly starshaped, we have

$$\lambda \frac{5-q}{2(q+1)} \int_{|u|\geq 1} |u|^{q+1} dx + \lambda \frac{5-p}{2(p+1)} \int_{|u|<1} |u|^{p+1} dx$$

$$+3\lambda \cdot \mu \left\{x \in \Omega \, |u| \geq 1\right\} \frac{q-p}{(q+1)(p+1)} = \frac{1}{2} \int_{\partial\Omega} (x, \nu) \left|\frac{\partial u}{\partial \nu}\right|^2 dx \geq \tag{17}$$

$$c \left(\textstyle\int_{\Omega} |\Delta u| \, dx\right)^2 .$$

We discuss separately the different cases.

(i) If $1 < q = p \leq 3$, the subcritical term g defined by (16) reduces to the case considered in Theorem 2.4 in [2], where the result is obtained by means of interpolation inequality and standard elliptic estimates. We omit the details.

(ii) If $3 \geq q > p > 1$, then $q - p > 0$ and

$$\|u\|_{q+1}^{q+1} \geq \int\limits_{|u|\geq 1} |u|^{q+1} dx > \mu \left\{x \in \Omega : |u| \geq 1\right\},$$

so that

$$\lambda c(p,q) \left(\|u\|_{q+1}^{q+1} + \|u\|_{p+1}^{p+1}\right) \geq c\|\Delta u\|_1^2. \tag{18}$$

Combining interpolation inequality (eventually the weak one, (15)) as in Theorem 2.4 in [2] we can obtain the following inequalities:

$$\|u\|_{q+1} \leq c\|\Delta u\|_1^{\frac{2}{q+1}}$$

$$\|u\|_{p+1} \leq c\|\Delta u\|_1^{\frac{2}{p+1}}$$

so that

$$\|u\|_{q+1}^{q+1} + \|u\|_{p+1}^{p+1} \leq c\|\Delta u\|_1^2 \leq \lambda c(p,q) \left(\|u\|_{q+1}^{q+1} + \|u\|_{p+1}^{p+1}\right);$$

this implies directly $\lambda \geq \lambda_0 > 0$.

(iii) If $5 > p > q > 1$, then $q - p < 0$ and (17) implies directly

$$\lambda c\,(q,p)\,\|u\|_{q+1}^{q+1} \geq c\,\|\Delta u\|_1^2\,,$$

where $q \in (1,3]$, so that we can conclude as in [2].

(iv) If $5 > p > 1 > q > 0$, (17) implies

$$\lambda c\,(q,p)\,\|u\|_2^2 \geq c\,\|\Delta u\|_1^2\,,$$

since $q - p < 0$; on the other hand, by standard elliptic estimates (10),

$$\|\triangle u\|_1^2 \geq c\,\|u\|_2^2\,,$$

so that

$$\lambda \geq \lambda_0.$$

(v) Finally, if $p \geq 5 > q$, (17) implies either

$$\lambda c\,(q,p)\,\|u\|_{q+1}^{q+1} \geq c\,\|\Delta u\|_1^2\,,$$

if $q \in (1, 3]$, or

$$\lambda c\,(q,p)\,\|u\|_2^2 \geq c\,\|\Delta u\|_1^2$$

if $q \in (0, 1)$. In both cases we can conclude as previously.

The proof of Theorem 1 is now complete.

3.2 The case $N = 4$.

We assume here that $N = 4$ and

$$g(u) = \begin{cases} |u|^{p-1} \cdot u & |u| < 1, \ \ 1 < p \\ |u|^{q-1} \cdot u & |u| \geq 1, \ \ 0 < q < 1 \end{cases} \tag{19}$$

Then we have the following result.

Theorem 2 *Let Ω be strictly starshaped about the origin; suppose that u is a solution of problem (1), with g given by (19). Then*

$$\lambda \geq \lambda_0(q, p, \Omega) > 0.$$

Proof of Theorem 2. By Pohozaev's identity (6), since Ω is strictly starshaped, we have

$$\begin{aligned} &\lambda \frac{3-q}{q+1} \int\limits_{|u|\geq 1} |u|^{q+1} dx + \lambda \frac{3-p}{p+1} \int\limits_{|u|<1} |u|^{p+1} dx \\ &+4\lambda \cdot \mu\left\{x \in \Omega : |u| \geq 1\right\} \frac{q-p}{(q+1)(p+1)} \geq c \left(\int\limits_{\Omega} |\triangle u| dx \right)^2 . \end{aligned} \tag{20}$$

Observe that $q - p < 0$ since $0 < q < 1 < p$, from (19). We discuss separately the different cases.

(i) If $1 < p < 2$, (20) implies

$$\lambda \frac{3-q}{q+1} \int\limits_{|u|\geq 1} |u|^{q+1} dx + \lambda \frac{3-p}{p+1} \int\limits_{|u|<1} |u|^{p+1} dx \geq c \,\|\triangle u\|_1^2 . \tag{21}$$

By potential theory,

$$u \le v = \frac{c}{|x|^2} * |\triangle u|$$

and $|x|^{-2} \in L^2_w$; this yields, by definition (12),

$$\|\triangle u\|_1^2 \ge c\|u\|_{2,w}^2. \tag{22}$$

Let us now consider separately the two integral terms in (21). On one hand, applying the weak interpolation inequality (15) we have

$$\int\limits_{|u|<1} |u|^{p+1} dx \le c\|\chi_{\{|u|<1\}} u\|_{2,w}^{(p+1)a} \|\chi_{\{|u|<1\}} u\|_3^{(1-a)(p+1)} \tag{23}$$

where $\frac{1}{p+1} = \frac{a}{2} + \frac{1-a}{3}$, that is, $a = \frac{4-2p}{p+1}$. Since $1 < p < 2$, (23) implies

$$\begin{aligned}\int\limits_{|u|<1} |u|^{p+1} dx &\le c\|u\|_{2,w}^{2(2-p)} \{ \int\limits_{|u|<1} |u|^3 \}^{p-1} \\ &\le c\|u\|_{2,w}^{2(2-p)} \{ \int\limits_{|u|<1} |u|^{p+1} \}^{p-1};\end{aligned}$$

then

$$\int\limits_{|u|<1} |u|^{p+1} dx \le c\|u\|_{2,w}^2. \tag{24}$$

On the other hand, by the equivalent definition of weak-L^2 norm, (11)

$$\begin{aligned}\int\limits_{|u|\ge 1} |u|^{q+1} dx &= \int\limits_{|u|\ge 1} dx \int_0^{|u|} (q+1)\alpha^q d\alpha \\ &= \int_0^1 (q+1)\alpha^q d\alpha \int\limits_{|u|\ge 1} dx + \int_1^\infty (q+1)\alpha^q d\alpha \int\limits_{|u|\ge\alpha} dx \\ &= \int_0^1 (q+1)\alpha^q \mu\{|u| \ge 1\} d\alpha + \int_1^\infty (q+1)\alpha^q \mu\{|u| \ge \alpha\} d\alpha \\ &\le \|u\|_{2,w}^2 + \|u\|_{2,w}^2 \int_1^\infty (q+1)\frac{1}{\alpha^{2-q}} d\alpha \\ &= \frac{2}{1-q}\|u\|_{2,w}^2. \end{aligned} \tag{25}$$

Combining (21), (22), (24) and (25) yields

$$\int\limits_{|u|\ge 1} |u|^{q+1}dx + \int\limits_{|u|<1} |u|^{p+1}dx \le c\|u\|^2_{2,w} \le \|\triangle u\|^2_1$$
$$\le c\lambda\{ \int\limits_{|u|\ge 1} |u|^{q+1}dx + \int\limits_{|u|<1} |u|^{p+1}dx\};$$

this implies directly $\lambda \ge \lambda_0 > 0$.

(ii) If $2 \le p < 3$, observe that

$$\lambda\frac{3-q}{q+1}\int\limits_{|u|\ge 1} |u|^{q+1}dx + \lambda\frac{3-p}{p+1}\int\limits_{|u|<1} |u|^{p+1}dx$$
$$\le \lambda c(q,p)\{ \int\limits_{|u|\ge 1} |u|^{q+1}dx + \int\limits_{|u|<1} |u|^{s+1}dx\}$$

with $s \in (q,p) \cap (1,2)$; hence we can repeat the proof given in previous point with s instead of p.

(iii) If $p \ge 3$, then (20) implies

$$\lambda\frac{3-q}{q+1}\int\limits_{|u|\ge 1} |u|^{q+1}dx \ge c\left(\int\limits_{\Omega} |\triangle u|dx\right)^2,$$

and we can conclude directly as in point (i).

The proof of Theorem 2 is now complete.

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