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PERTURBATION FROM SYMMETRY AND MULTIPLICITY OF SOLUTIONS FOR ELLIPTIC PROBLEMS WITH SUBCRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^2$

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Abstract. We consider the following boundary value problem

$$\begin{cases} -\Delta u = g(x,u) + f(x,u) & x \in \Omega \\ \\ u = 0 & x \in \partial\Omega \end{cases}$$

where $g(x,-\xi) = -g(x,\xi)$ and g has subcritical exponential growth in $\mathbb{R}^2$. Using the method developed by Bolle, we prove that this problem has infinitely many solutions under suitable conditions on the growth of $g(u)$ and $f(u)$.

1. **Introduction.** In the last few years, many authors have widely investigated existence and multiplicity of solutions for semilinear elliptic problems with Dirichlet boundary conditions by using variational methods and topological arguments (see [11] and references therein). In particular, the following model

$$\begin{cases} -\Delta u = |u|^{p-2}u + f(x) & x \in \Omega \\ \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

has been extensively studied, where Ω is an open bounded domain of $\mathbb{R}^N$, $N \geq 3$, $f \in L^2(\Omega)$ and $2 < p < 2N/(N-2)$. If $f \equiv 0$, equation (1) possesses a natural $\mathbb{Z}_2$-symmetry, which guarantees the existence of an unbounded sequence of critical values for the symmetric functional associated to the problem. On the other hand, if $f \neq 0$ the problem loses its $\mathbb{Z}_2$ symmetry and a natural question is whether the infinite number of solutions is preserved under perturbation; a partial answer was independently obtained by Struwe [12], Bahri-Berestycki [2], Rabinowitz [9] and Bahri-Lions [3], who showed in important works that the multiplicity structure can be maintained also in the perturbed case, restricting the growth range of the nonlinearity with suitable bounds depending on N. The main idea is to think of the non-symmetric functional I under study as a perturbation of its symmetric part I_0

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and then to estimate how the growth rate of the critical levels of I_0 is affected by estimating the perturbation $I - I_0$.

More recently, a new type of perturbation from symmetry has been considered, resulting from second order systems with non-homogeneous boundary conditions: if $f = 0$ in (1) but $u|_{\partial\Omega} = u_0 \neq 0$, the symmetry is again broken and the perturbation - due to the non-homogeneous boundary condition - is of higher order. The standard perturbative method can be applied but yields a result for an even smaller range of p values. It was to deal with this type of perturbation that Bolle [4] developed his new approach: this new method deals with I as the end-point of a continuous path of functionals I_θ, $\theta \in [0, 1]$, which starts at the symmetric functional I_0. Bolle's abstract theorem states, roughly speaking, that the preservation of the min-max critical levels along the path of functionals I_θ depends only on the velocities of deformation $\frac{\partial}{\partial\theta} I_\theta(u)$ at the critical points u of I_θ. This fact often allows to obtain better estimates at such points since they obey certain conservation laws, being solutions of the corresponding Euler-Lagrange equations. Bolle, Ghoussoub and Tehrani [5] tested this approach on several other problems, including the non-homogeneous problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & x \in \Omega \\ \\ u = u_0 & x \in \partial\Omega, \end{cases}$$

proving the existence of infinitely many solutions for a larger range, namely for $1 < p < (N+1)/(N-1)$. Later, Chambers and Ghoussoub [6] have applied Bolle's approach to establish a general multiplicity result for problems with broken symmetry, where the forcing term f depends also on u; they have been able to prove that the infinite sequence of critical values is preserved if p belongs to a range of values depending also on the growth of $f(u)$: roughly speaking, the lower is the growth of the perturbation $f(u)$, the better is the bound for p obtained.

In this paper we deal with an analogue of problem (1) in dimension $N = 2$. Let Ω be an open bounded subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$; we are concerned with existence and multiplicity results for nonlinear elliptic equations of the type

$$\begin{cases} -\Delta u = g(x, u) + f(x, u) & x \in \Omega \\ \\ u = 0 & x \in \partial\Omega \end{cases} \tag{2}$$

where $g(x, -\xi) = -g(x, \xi)$ and g has subcritical growth in $\mathbb{R}^2$. When $N = 2$, the notion of criticality, that is, the maximal growth on u which allows to treat problem (2) variationally, is motivated by the so called Trudinger-Moser inequality [8], [15], [7], which says that for $\alpha \leq 4\pi$

$$\sup_{\|u\|_{H_0^1} \leq 1} \int_\Omega e^{\alpha u^2} \leq c(\alpha)|\Omega| \leq c(4\pi)|\Omega| = C_{TM}|\Omega|$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and C_{TM} is a constant which does not depend on u; hence, the maximal growth permitted to study problem (2) variationally is of exponential type. Motivated by the Trudinger- Moser inequality, we say that g has subcritical growth at $+\infty$ if for all $\alpha > 0$

$$\lim_{t \to +\infty} \frac{|g(t)|}{e^{\alpha t^2}} = 0.$$

To our knowledge, the problem of perturbation from symmetry equations with exponential growth in a bounded domain of $\mathbb{R}^2$ has been approached only by Sugimura [13], who proved that an infinite number of solutions are preserved if the nonlinear term has an exponential growth of the kind $e^{|t|^q}, 0 < q < 1/2$, and the forcing term $f = f(x)$ does not depend on u. In this paper we approach the problem (2) using Bolle's method: following the idea in [CG], we are able to extend the result of Sugimura to a perturbed problem with forcing term depending also on u. As just remarked, the maximal growth allowed depends now on the growth of $f(u)$: in particular we prove the existence of infinite solutions for (2) if, roughly speaking, $g(t) \sim e^{|t|^q}, 0 < q < 1$ and $f(u)$ satisfies suitable growth's conditions. Our result includes the one obtained by Sugimura as a special case.
More precisely, in this paper we obtain multiplicity results for perturbed problems of the following type:

$$\begin{cases} -\Delta u = g\left(x,u\right) + f\left(\theta,x,u\right) & x \in \Omega \\ \\ u = 0 & x \in \partial\Omega \end{cases}$$

where $g(x,\cdot)$ is odd, has exponential growth (as will be defined) and f is a non-symmetric perturbative term. The precise statement of our main result is given in Section 3, Theorem 4. As an application of our result, let us consider the following model problem, which presents the main features of the general case:

$$\begin{cases} -\Delta u = ue^{|u|^q} + f(x)|u|^{r-1} & \text{on } \Omega \\ \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where

$$q > 0, \qquad r \geq 0,$$

and $f \in L^s(\bar{\Omega})$ for some $s > 1$. Then we have:

Theorem 1. *Let*

$$0 < q < 1, \qquad 0 \leq r < 2 - 2q. \tag{4}$$

Then equation (3) possesses an unbounded sequence of weak solutions in $H_0^1(\Omega)$.

Remark 1. *For* $0 < r < 1$ *the perturbative term* $f(x)|u|^{r-1}$ *has to be modified in* $f(x)(|u|+1)^{r-1}$.

As just observed, Theorem 1.1 includes Sugimura's result as $r = 1$. Note that (4) allows $q \in [0,1)$, depending on the value of r: this depending of the perturbation term on u allows the exponent q to go over the value $\frac{1}{2}$.

Remark 2. *As in* $\mathbb{N} \geq 3$, *it remains an open problem, whether the result continues to hold up to the critical growth, i.e., up to the Trudinger-Moser critical exponent* $q = 2$.

2. **Preservation of critical levels under deformation of an even functional.**

In this section we recall Bolle's method for dealing with problems with deformation from symmetry (see e.g. [5], [6]). Consider two continuous functions ρ_1, ρ_2 : $[0,1] \times \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous relative to the second variable.

Assume $\rho_1 \leq \rho_2$ and denote by ψ_1, ψ_2 the scalar fields associated to them, defined on $[0,1] \times \mathbb{R}$ by

$$\begin{cases} \psi_i(0,s) = s \\ \\ \frac{\partial}{\partial\theta}\psi_i(\theta,s) = \rho_i(\theta,\psi_i(\theta,s)). \end{cases}$$

Note that ψ_1 and ψ_2 are continuous in s and that for all $\theta \in [0,1]$, $\psi_1(\theta,\cdot)$ and $\psi_2(\theta,\cdot)$ are non decreasing on $\mathbb{R}$; moreover, since $\rho_1 \leq \rho_2$ we have $\psi_1 \leq \psi_2$.
Let E be a Hilbert space (with scalar product $\langle,\rangle$ and associated norm $\|\cdot\|$) and consider a $\mathcal{C}^2$ functional I_0 on E; let I be a $\mathcal{C}^2$ functional : $[0,1] \times E \to \mathbb{R}$. For subsets $U \subset V$ of E, denote by

$$c_U(\theta) = \sup_U I_\theta, \quad c_{V,U}(\theta) = \inf_{g \in S_{V,U}} \sup_{g(U)} I_\theta$$

where $I_\theta = I(\theta,\cdot)$

$S_{V,U} = \{g \in \mathcal{C}(V,E) : g(u) = u \text{ for } u \in U \text{ and } g(u) = u \text{ for } \|u\| > R, \text{for some } R > 0\}$.

We make the following assumptions:

(H1) *I satisfies a kind of Palais-Smale condition:* for every sequence (θ_n, u_n) in $[0,1] \times E$ such that $I(\theta_n,u_n)$ is bounded and $\lim_{n\to\infty} \|I'_{\theta_n}(u_n)\| = 0$ there is a subsequence converging in $[0,1] \times E$. The limit (θ,u) then satisfies $I'_\theta(u) = 0$.

(H2) For all $b > 0$ there is a constant $C_1(b)$ such that:

$$|I_\theta(u)| < b \quad \text{implies} \quad |\frac{\partial I}{\partial\theta}(\theta,u)| \leq C_1(b)(\|I'_\theta(u)\| + 1)(\|u\| + 1).$$

(H3) For all critical points u of I_θ,

$$\rho_1(\theta, I_\theta(u)) \leq \frac{\partial I}{\partial\theta}(\theta,u) \leq \rho_2(\theta, I_\theta(u))$$

(H4) There are two closed subsets B and $A \subset B$ of E such that
   (i) I_0 has an upper bound on B and for some $\theta_0 \in [0,1]$

$$\lim_{\|u\|\to+\infty, u \in B} \sup_{\theta \in [0,\theta_0]} I_\theta(u) = -\infty.$$

   (ii) $c = c_{B,A}(0) > b = c_A(0)$
   In the sequel we will say that I_0 *has a min-max configuration*(c,b) if it satisfies hypothesis (H4).

(H4') I_0 is even and for any finite dimensional subspace W of E and any θ we have

$$\lim_{\|w\|\to\infty, w \in W} \sup_{\beta \in [0,\theta]} I(\beta,w) = -\infty.$$

In the sequel we will say that $I(\theta,u)$ is *a good path of functionals* if it satisfies hypotheses (H1), (H2), (H3) and(H4').
Observe that we are assuming implicitly in the above conditions that the starting functional I_0 satisfies the Palais-Smale condition (H_1) and (H_4) for $\theta = 0$. Set

$$\bar{\rho}_1(\theta,t) = \sup_{\beta \in [0,\theta]} \rho_1(\beta,t), \qquad \bar{\rho}_2(\theta,t) = \sup_{\beta \in [0,\theta]} \rho_2(\beta,t).$$

The main idea of Bolle's result is the following: If one assumes a min-max critical level for the initial functional I_0, then the deformation velocities ρ_1 and ρ_2 will determine whether this critical level persists along the path. We are now ready to present a reformulation of Bolle's result due to Chambers and Ghoussoub (see [6] for further references and [4] for the original result)

Theorem 2. *Let* I_0 *be a* $\mathcal{C}^2$*- functional on* E *with a min-max configuration* (c, b)*, as defined in* $(H4)$*; let* $\rho_1 \leq \rho_2$ *be two velocity fields and* ψ_1*,* ψ_2 *be the corresponding scalar flows. If* $\psi_1(\theta_0, c_{B,A}(0)) > \psi_2(\theta_0, c_A(0))$ *for some* $\theta_0 \in [0,1]$ *then for any path of functionals* $I : [0,1] \times E \to \mathbb{R}$ *satisfying* $(H1)$*,* $(H2)$*,* $(H3)$ *and* $(H4')$ *the functional* I_{θ_0} *has a critical point at a level* $\bar{c}$ *such that:*

$$\psi_1(\theta_0, c_{B,A}(0)) \leq \bar{c} \leq \psi_2(\theta_0, c_{B,A}(0)).$$

Note that if $c = c_{B,A}(0) > b = c_A(0)$, as assumed in (ii) of $(H4)$, it is standard to show that the functional I_0 has a critical point at level c: Bolle's theorem assures that this min-max critical level is preserved along any path of functionals satisfying the above hypotheses if $\psi_1(\theta_0, c_{B,A}(0)) > \psi_2(\theta_0, c_A(0))$. Assume now that the Hilbert space E is decomposed as

$$E = \overline{\bigcup_{k=0}^{\infty} E_k} \quad \text{where } E_k = E_{k-1} \oplus \mathbb{R}e_k \tag{5}$$

with $E_0 = E_-$ being a finite dimensional subspace and $(E_k)_{k=1}^{\infty}$ is an increasing sequence of subspaces of E.
Let us set

$$\mathcal{G} = \{g \in \mathcal{C}(E, E) : g \text{ is odd and for a fixed } R > 0 \;\; g(u) = u \text{ for } \|u\| \geq R\}$$

and

$$c_k = \inf_{g \in \mathcal{G}} \sup_{u \in g(E_k)} I_0(u). \tag{6}$$

In this framework, the following abstract result can be proved (see [6]).

Lemma 1. *Let* $\rho_1 \leq \rho_2$ *be two velocity fields and let* ψ_1, ψ_2 *be the corresponding scalar flows. Let* I_0 *be an even* $\mathcal{C}^2$ *functional on* E *and consider the levels* c_k *associated to* I_0 *defined by (6). Then there is* $C > 0$*, depending only on* ρ_1, ρ_2 *such that for every* $k \in \mathbb{N}$ *and every* $\theta \in [0,1]$*:*
either

(i) $\psi_2(\theta, c_k) < \psi_1(\theta, c_k)$,

or

(ii) $c_{k+1} - c_k \leq C\theta[\bar{\rho}_1(\theta, c_{k+1}) + \bar{\rho}_2(\theta, c_k) + 1]$.

Moreover, in case (i) *there exists a level* $\ell_k(\theta)$*, only depending on* I_0 *and* ρ_1, ρ_2*, such that for any "good" path of functionals* I *satisfying hypotheses* $(H1)$*,* $(H2)$*,* $(H3)$ *and* $(H4')$ *there exists a critical level* $\bar{c}_k(\theta)$ *for* I_θ *with* $\psi_2(\theta, c_k) < \psi_1(\theta, c_{k+1}) \leq \bar{c}_k(\theta) \leq \ell_k(\theta)$*.*

We are now ready to prove the main abstract result of this paper

Theorem 3. *Let* $\rho_1 \leq \rho_2$ *be two velocity fields and let* ψ_1, ψ_2 *be the corresponding scalar flows. Assume now that the Hilbert space* E *is decomposed as in (5). Let* I_0 *be an even* $\mathcal{C}^2$ *functional on* E *and consider the levels* c_k *associated to* I_0 *defined by (6). We have :*

(a) *if* $\psi_1(\theta, c_k) \uparrow +\infty$ *as* $k \to \infty$*, then for every* $N \in \mathbb{N}$ *there exists a* $\theta_N \in (0,1]$*, depending only on* I_0 *and* ρ_1, ρ_2*, such that for any good path of functionals* $I : [0,1] \times E \to \mathbb{R}$ *satisfying* $(H1)$*,* $(H2)$*,* $(H3)$ *and* $(H4')$ *the functional* I_θ *has at least* N *distinct critical levels, for any* $\theta \in [0, \theta_N]$*;*

(b) *if* $c_k \geq B_1 + B_2 k(\ln k)^{\bar{\beta}}$ *where* $\bar{\beta} > 0, B_1 \in \mathbb{R}, B_2 > 0$ *and if* $\bar{\rho}_i(\theta, t) \leq A_1 + A_2(\ln(|t|+1))^{\bar{\alpha}}$ *where* $\bar{\alpha} \geq 0$ *and* $A_1, A_2 \geq 0$*, then* I_1 *has an unbounded sequence of critical levels provided* $\bar{\beta} > \bar{\alpha}$*.*

Proof of Theorem 3. Theorem 3 is a consequence of Lemma 1:

(a) Our aim is to prove that for any $N \in \mathbb{N}$ and for any good path of functional I_θ, there is a $\theta_N \in (0,1]$ such that the functionals I_θ have N distinct critical levels $d_1(\theta) < d_2(\theta) < ... < d_N(\theta)$, for every $\theta \in [0, \theta_N]$. We obtain the desired sequence of N critical levels by induction. Let $C > 0$ denote the constant appearing in Lemma 1; define

$$\eta_k = \inf\{\theta \in [0,1] : c_{k+1} - c_k \leq C\theta[\bar{\rho}_1(\theta, c_{k+1}) + \bar{\rho}_2(\theta, c_k) + 1]\}.$$

Since $\psi_1(\theta, c_k) \uparrow +\infty$ as $k \to +\infty$ by assumption, the sequence c_k is unbounded and for any $M > 0$ there is a $K_M > 0$ such that

$$\psi_1(\theta, c_{k+1}) > M \quad \text{for all } \theta \in [0,1] \text{ and } k > K_M;$$

let us fix $M_1 = 1$, $k_1 = K_{M_1} + 1$ and $\theta_1 = \eta_{k_1}/2$. By the definition of θ_1, for all $\theta \in [0, \theta_1]$ the alternative (ii) of Lemma 1 is not valid; therefore (i) holds, so that for any $\theta \in [0, \theta_1]$ and any path of functionals $I : [0,1] \times E \to \mathbb{R}$ satisfying $(H1)$, $(H2)$, $(H3)$ and $(H4')$, the functionals I_θ have critical values $d_1(\theta)$ with

$$1 = M_1 < \psi_2(\theta, c_{k_1}) < \psi_1(\theta, c_{k_1+1}) < d_1\theta) \leq \ell_{k_1}(\theta).$$

Let now $N \in \mathbb{N}$ and suppose that there is a $\theta_{N-1} \in (0,1]$ such that for any path of functionals (satisfying the hypotheses of Lemma 1) the functionals I_θ have critical values $d_1(\theta) < d_2(\theta) < ... < d_{N-1}(\theta) \leq \ell_{k_{N-1}}(\theta)$. Let

$$M_N > \sup_{\theta \in [0, \theta_{N-1}]} \ell_{k_{N-1}}(\theta)$$

and let $K_N \in \mathbb{N}$ such that

$$\psi_1(\theta, c_{k+1}) > M_N \quad \text{for all} \quad k > K_N, \ \ \theta \in [0,1],$$

which exists by assumption; define $k_N = K_N + 1$ and $\theta_N = \eta_{k_N}/2$. Again, it is clear that for any $\theta \in [0, \theta_N]$ and any good family of functionals $I : [0,1] \times E \to \mathbb{R}$, the functionals I_θ have critical values $d_N(\theta)$ with

$$M_N < \psi_2(\theta, c_{k_N}) < \psi_1(\theta, c_{k_N+1}) < d_N(\theta) \leq \ell_{k_N}(\theta);$$

but, by definition, $M_N > \sup_{\theta \in [0, \theta_{N-1}]} \ell_{k_{N-1}}(\theta)$; hence, by hypothesis of induction, we can conclude that the functionals I_θ have N distinct critical values satisfying $d_1(\theta) < d_2(\theta) < ... < d_N(\theta)$, that is (a).

(b) Let us suppose by contradiction that the functional I_1 has only finitely many critical levels. Since ρ_i is Lipschitz continuous in the second variable, then there are $L_i > 0$ such that $|\psi_i(\theta, s) - s| \leq \theta(\bar{\rho}_i(\theta, s) + L_i)$; hence

$$\psi_1(1, s) \geq s - \bar{\rho}_1(1, s) - L_1 \geq s - A_3 - A_2(\ln(|s|+1))^{\bar{\alpha}}. \tag{7}$$

Since c_k is unbounded, (7) implies that also $\psi_1(1, c_{k+1})$ is unbounded; therefore, if we suppose that I_1 has only finitely many critical points, the alternative (i) of Lemma 1 can not hold, so that (ii) must be true (with $\theta = 1$). Then we have

$$\begin{aligned} c_{k+1} - c_k &\leq C(\bar{\rho}_1(1, c_{k+1}) + \bar{\rho}_2(1, c_k) + 1) \\ &\leq C(A_1 + A_2(\ln(|c_{k+1}|+1))^{\bar{\alpha}} + A_1 + A_2(\ln(|c_k|+1))^{\bar{\alpha}} + 1) \end{aligned}$$

which implies that there is a $k_1 > 0$ such that

$$c_{k+1} \leq c_k + A_4(\ln c_k)^{\bar{\alpha}} \quad \text{for all} \;\; k > k_1. \tag{8}$$

Starting from this relation, one can obtain

$$c_k < C_1 k(\ln k)^{\bar{\alpha}} \quad \text{for all} \quad k > k_1. \tag{9}$$

Estimate (9) has been proved by Sugimura [13], hence we will be brief; let us choose C_1 such that (9) is verified for $k = k_1$, and

$$\frac{A_4}{C_1}\left(1 + \bar{\alpha} + \ln C_1\right)^{\bar{\alpha}} \leq 1;$$

assume now that for $k > k_1$ (9) is valid. Then, by the choice of C_1

$$\begin{aligned} c_{k+1} &\leq c_k + A_4(\ln c_k)^{\bar{\alpha}} \\ &\leq C_1 k(\ln k)^{\bar{\alpha}} + A_4\left[\ln C_1 + \ln k + \bar{\alpha}\ln\ln k\right]^{\bar{\alpha}} \\ &\leq C_1(\ln(k+1))^{\bar{\alpha}}\left[k + \frac{A_4}{C_1}\left(\ln C_1 + 1 + \bar{\alpha}\right)^{\bar{\alpha}}\right] \\ &\leq C_1(k+1)(\ln(k+1))^{\bar{\alpha}}, \end{aligned}$$

that is (9) for $k+1$.

Now, we recall that, by assumption, $c_k \geq B_1 + B_2 k(\log k)^{\bar{\beta}}$, which is a contradiction under the further assumption that $\bar{\beta} > \bar{\alpha}$. ☐

3. Perturbation of a symmetric elliptic problem with exponential growth.

The aim of this section is to prove the existence of infinitely many solutions for the following perturbed elliptic problem

$$\begin{cases} -\Delta u = g\left(x,u\right) + f\left(\theta,x,u\right) & x \in \Omega \\ \\ u = 0 & x \in \partial\Omega \end{cases} \tag{10}$$

where $g(x,\cdot)$ is odd, has exponential growth (as will be defined) and f is a perturbative term. Let us define $G(x,\xi) = \int_0^\xi g(x,t)dt$ and $F(\theta,x,\xi) = \int_0^\xi f(\theta,x,t)dt$. We make the following standard assumptions on the symmetric term g (see also [13]) :

(g1) $g \in \mathcal{C}(\bar{\Omega}\times\mathbb{R},\mathbb{R})$;

(g2) there is a constant $A_0 > 0$ such that

$$|g(x,\xi)| \leq A_0 e^{\phi(\xi)} \quad \text{for} \quad (x,\xi) \in \bar{\Omega}\times\mathbb{R},$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\phi(\xi)\xi^{-2} \to 0$ as $|\xi| \to +\infty$;

(g3) there are constants $\mu > 0$ and $r_0 \geq 0$ such that $0 < G(x,\xi)\ln G(x,\xi) \leq \mu\xi g(x,\xi)$ for $x \in \bar{\Omega}$ and $|\xi| \geq r_0$;

(g4) $g(x,-\xi) = -g(x,\xi)$ for $(x,\xi) \in \bar{\Omega}\times\mathbb{R}$

(g5) there exist $0 < \alpha_1 \leq \alpha_2 < 1, A_1, A_2 > 0$, and $B_1, B_2 > 0$ such that

$$A_1 e^{|\xi|^{\alpha_1}} - B_1 \leq G(x,\xi) \leq A_2 e^{|\xi|^{\alpha_2}} + B_2 \quad \text{for} \quad (x,\xi) \in \bar{\Omega}\times\mathbb{R},$$

and we make the following assumptions on the perturbative term f

(f1) $f \in \mathcal{C}([0,1]\times\bar{\Omega}\times\mathbb{R},\mathbb{R})$ and $f(0,\cdot,\cdot) = 0$;

(f2) there are $r > 0$ and $c_1, c_2 > 0$ such that

$$\left|\frac{\partial}{\partial\theta}F(\theta,x,\xi)\right| \leq c_1|\xi|^r + c_2$$

(f3) there are $c_3 > 0$ and $\varphi(x) \in L^s(\Omega)$ for some $s > 1$ such that

$$|f(\theta, x, \xi)| \leq c_3 \left(|\xi| + 1\right)^{r-1} + \varphi(x);$$

Using the above notation, we have

Theorem 4. *Suppose that g satisfies (g1)-(g5), and suppose that the perturbative term f satisfies (f1)-(f3). Then we have:*

- (i) *for every $N \in \mathbb{N}$ there exists a $\theta_N \in (0, 1]$ such that problem (10) has at least N distinct solutions;*
- (ii) *if in addition $2/\alpha_2 - 2 > r/\alpha_1$, then problem (10) has an infinite number of solutions for all $\theta \in [0, 1]$.*

Proof of Theorem 4. Let $E = H_0^1(\Omega)$ be the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2},$$

and define the path of functionals $I : [0, 1] \times E \to \mathbb{R}$ by

$$I_\theta(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 - G(x, u) - F(\theta, x, u))dx;$$

then I_0 is an even functional and the critical points of I_θ are solutions of problem (10). The Palais Smale condition $(H1)$ is verified, as will be proved in Lemma 3 below. Lemma 2 below assures that condition $(H2)$ is satisfied. It is also easy to show that condition $(H4')$ is satisfied in any finite dimensional subspace of E, thanks to the exponential growth of the nonlinear term $G(x, u)$.
For each k, denote by E_k the subspace of E spanned by the first k eigenfunctions of $\triangle$; then, let

$$\mathcal{G} = \{g \in \mathcal{C}(E, E) : g \text{ is odd and } g(x) = x \text{ for large } \|x\|\},$$

and set $c_k = \inf_{g\in\mathcal{G}} \sup_{g(E_k)} I_0$. As before, c_k are critical levels of the even functional I_0 . Lemma 4 below shows that $(H3)$ holds with

$$\begin{aligned} \rho_1(\theta, s) &= -C(\ln(|s| + 1))^{\frac{r}{\alpha_1}}, \\ \rho_2(\theta, s) &= C(\ln(|s| + 1))^{\frac{r}{\alpha_1}}; \end{aligned}$$

on the other hand, it is shown by Sugimura [13] that there are positive constants B_1, B_2 such that $c_k \geq B_1 k(\ln k)^{\frac{2}{\alpha_2}-2} - B_2$. Therefore we can apply Theorem 3 (b) with $\bar{\alpha} = \frac{r}{\alpha_1}$ and $\bar{\beta} = \frac{2}{\alpha_2} - 2$ to obtain that I_1 has an infinite number of solutions when $\frac{2}{\alpha_2} - 2 > \frac{r}{\alpha_1}$, which is the claim in Theorem 4 (ii).
Theorem 4 (i) follows directly from Theorem 3 (a) , since $\psi_1(\theta, c_k) \uparrow +\infty$ as $k \to \infty$ for the ρ_i defined above. □

It remains to show the following lemmas:

Lemma 2. *For all $b > 0$ there is a constant $C_1(b)$ such that*
$|I_\theta(u)| < b$ implies

$$|\frac{\partial I}{\partial \theta}(\theta, u)| \leq C_1(b)(\|I'_\theta(u)\| + 1)(\|u\| + 1).$$

Proof of Lemma 2. By $(g5)$ and $(f3)$ we have

$$\begin{aligned} G(x,u)+F(\theta,x,u) &\geq A_1 e^{|u|^{\alpha_1}} - B_1 - \frac{c_3}{r}(|u|+1)^r - \frac{c_3}{r} - |u||\varphi(x)| \\ &\geq \widetilde{A}_1 e^{|u|^{\alpha_1}} - \widetilde{B}_1 \quad \text{if} \quad r>0 \end{aligned}$$

and

$$\begin{aligned} G(x,u)+F(\theta,x,u) &\geq A_1 e^{|u|^{\alpha_1}} - B_1 - c_3 \ln(|u|+1) - |u||\varphi(x)| \\ &\geq \widetilde{A}_1 e^{|u|^{\alpha_1}} - \widetilde{B}_1 \quad \text{if} \quad r=0, \end{aligned}$$

where $\widetilde{A}_1, \widetilde{B}_1$ can denote different constants.
Let $b>0$ and suppose that $|I_\theta(u)|<b$; by definition of I_θ, $\int_\Omega (G(x,u)+F(\theta,x,u))dx = \frac{1}{2}\|u\|^2 - I_\theta(u)$, so that

$$\begin{aligned} \widetilde{A}_1 \int_\Omega e^{|u|^{\alpha_1}} dx &\leq \int_\Omega G(x,u)dx + \int_\Omega F(\theta,x,u)dx + \widetilde{B}_1|\Omega| \\ &\leq \frac{1}{2}\|u\|^2 + |I_\theta(u)| + \widetilde{B}_1|\Omega| \\ &\leq \frac{1}{2}\|u\|^2 + b + \widetilde{B}_1|\Omega|. \end{aligned} \tag{11}$$

Combining $(g3)$, $(f3)$ with (11), and denoting s' the conjugate exponent of s, yields

$$\begin{aligned} -\langle I'_\theta(u),u\rangle &= \int_\Omega (-|\nabla u|^2 + g(x,u)u + f(\theta,x,u)u)dx \\ &\geq \int_\Omega |\nabla u|^2 dx + \int_\Omega (g(x,u)u - 4G(x,u))dx \\ &\quad + \int_\Omega (f(\theta,x,u)u - 4F(\theta,x,u))dx - 4I_\theta(u) \\ &\geq \|u\|^2 - A_5 - A_6 \int_\Omega |u|^r dx - A_7\|\varphi\|_s\|u\|_{s'} - 4b \\ &\geq \|u\|^2 - \widetilde{A}_1 \int_\Omega e^{|u|^{\alpha_1}} dx - A_8\|\varphi\|_s\|u\| - A_9 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\|u\|^2 - A_{10} \\ &= \frac{1}{4}\|u\|^2 - A_{10}. \end{aligned}$$

Therefore,

$$\|I'_\theta(u)\|\|u\| \geq -\langle I'_\theta(u),u\rangle \geq \frac{1}{4}\|u\|^2 - A_{10}; \tag{12}$$

finally, combining $(f2)$ with (12) and (11)

$$\begin{aligned}
|\frac{\partial}{\partial\theta}I(\theta,u)| &= |\int_\Omega \frac{\partial}{\partial\theta}F(\theta,x,u)dx| \\
&\leq c_1\int_\Omega |u|^r dx + c_2|\Omega| \\
&\leq \frac{1}{2}\int_\Omega e^{|u|^{\alpha_1}}dx + A_{11} \\
&\leq \frac{1}{4}\|u\|^2 + A_{12} \\
&\leq \|I'_\theta(u)\|\|u\| + A_{13} \\
&\leq C(\|I'_\theta(u)\|+1)(\|u\|+1).
\end{aligned}$$

□

Lemma 3. *For every sequence* (θ_n,u_n) *in* $[0,1]\times E$ *such that*

$$|I(\theta_n,u_n)| \quad \textit{is bounded}$$

$$\lim_{n\to\infty}\|I'_{\theta_n}(u_n)\| = 0$$

there is a subsequence converging in $[0,1]\times E$.

Proof of Lemma 3. The proof of this lemma is standard, so we will be brief. Assume that $\{(\theta_n,u_n)\}$ in $[0,1]\times E$ satisfies $I(\theta_n,u_n)\leq M$ and $\|I'(\theta_n,u_n)\|_{E^*}\to 0$ as $n\to+\infty$. Then for all large n and $\rho>0$, using (11), $(g3)$ and $(f3)$, we have

$$\begin{aligned}
M+\rho\|u_n\| &\geq I(\theta_n,u_n)-\rho\langle I'(\theta_n,u_n),u_n\rangle \\
&= (\frac{1}{2}-\rho)\|u_n\|^2+\int_\Omega(\rho g(\theta_n,x,u_n)u_n-G(\theta_n,x,u_n))dx \\
&\quad +\int_\Omega(\rho f(\theta_n,x,u_n)u_n-F(\theta_n,x,u_n))dx \\
&\geq (\frac{1}{2}-\rho)\|u_n\|^2+C\int_\Omega(G(\theta_n,x,u_n)\ln(G(\theta_n,x,u_n))dx \\
&\quad -C\|u_n\|_r^r-C\|\varphi\|_s\|u_n\|_{s'}-C \\
&\geq (\frac{1}{2}-\rho)\|u_n\|^2-\frac{1}{4}\int_\Omega e^{|u|^{\alpha_1}}dx-C\|u_n\|-C \\
&\geq (\frac{1}{4}-\rho)\|u_n\|^2-C\|u_n\|-C,
\end{aligned}$$

where C stands for different constants; this implies that $\{(\theta_n,u_n)\}$ is bounded in $[0,1]\times E$. On the other hand, we see that $I'(\theta_n,u_n)$ is of the following form

$$I'(\theta_n,u_n)=Lu_n-P(\theta_n,u_n)$$

where $L:E\to E^*$ is a fixed boundedly invertible linear map and $P(\cdot,u):E\to E^*$ is compact. By boundedness of $\{(\theta_n,u_n)\}$ and compactness of P the sequences $L^{-1}P(\theta_n,u_n)$, and hence $\{(\theta_n,u_n)\}$, are relatively compact in E. □

Lemma 4. *There exists a constant* $C>0$ *such that if* $u\in H_0^1(\Omega)$ *is a critical point of* I_θ*, then*

$$|\frac{\partial I}{\partial\theta}(\theta,u)|\leq C(\ln(|I_\theta(u)|+1))^{\frac{r}{\alpha_1}}.$$

Proof of Lemma 4. For simplicity, in the sequel C stands for different constants. Let us suppose that $I'_\theta(u) = 0$; then, combining $(g3)$, $(g5)$, $(f3)$ yields

$$\begin{aligned} I_\theta(u) &= I_\theta(u) - \frac{1}{2}\langle I'_\theta(u), u\rangle \\ &= \int_\Omega (\frac{1}{2}g(x,u)u - G(x,u))dx + \int_\Omega (\frac{1}{2}f(\theta,x,u)u - F(\theta,x,u))dx \\ &\geq (\frac{1}{2\mu} - \varepsilon)\int_\Omega G(x,u)\ln G(x,u)dx - C\|u\|_r^r - C\|\varphi\|_s\|u\|_{s'} - C \\ &\geq C\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx - C\|u\|_{s'} - C. \end{aligned} \tag{13}$$

Now observe that the function $\varphi(t) = t^{\alpha_1/p}e^{t^{\alpha_1/p}}$ is convex for $t \geq t(p,\alpha_1) = t_p$, for any $p > 0$; for now on let choose $p = r$, where r is the exponent appearing in $(f2)$ and $(f3)$. Applying Young's inequality with $t = |u|^r$ we obtain

$$\int_{|u|>t_r^{1/r}} |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx \geq |\Omega|\{\int_{|u|>t_r^{1/r}} |u|^r dx\}^{\frac{\alpha_1}{r}} e^{\frac{1}{|\Omega|}\{\int_{|u|>t_r^{1/r}} |u|^r dx\}^{\frac{\alpha_1}{r}}}$$

where $|\Omega|$ is the Lebesgue measure of Ω; then, since $\|u\|_r \leq C + \{\int_{|u|>t_r^{1/r}} |u|^r dx\}^{\frac{1}{r}}$,

$$\begin{aligned} \|u\|_r^{\alpha_1}e^{\frac{1}{|\Omega|}\|u\|_r^{\alpha_1}} &\leq \{C + C(\int_{|u|>t_r^{1/r}} |u|^r dx)^{\frac{\alpha_1}{r}}\}e^{\frac{1}{|\Omega|}(C + C\{\int_{|u|>t_r^{1/r}} |u|^r dx\}^{\frac{\alpha_1}{r}})} \\ &\leq C + C\{\int_{|u|>t_r^{1/r}} |u|^r dx\}^{\frac{\alpha_1}{r}} e^{\frac{1}{|\Omega|}\{\int_{|u|>t_r^{1/r}} |u|^r dx\}^{\frac{\alpha_1}{r}}} \\ &\leq C + C\int_{|u|>t_r^{1/r}} |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx \\ &\leq C + C\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx. \end{aligned} \tag{14}$$

In the same way, one can also prove that

$$\|u\|_{s'}^{\alpha_1}e^{\frac{1}{|\Omega|}\|u\|_{s'}^{\alpha_1}} \leq C + C\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx,$$

which implies directly

$$\|u\|_{s'} \leq C\{\ln(\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx + 1)\}^{\frac{1}{\alpha_1}} + C. \tag{15}$$

Combining (14), (15) and (13) yields

$$\begin{aligned} I_\theta(u) &\geq C\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx - \{\ln(\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx + 1)\}^{\frac{1}{\alpha_1}} - C \\ &\geq C\int_\Omega |u|^{\alpha_1}e^{|u|^{\alpha_1}}dx - C \\ &\geq C\|u\|_r^{\alpha_1}e^{\frac{1}{|\Omega|}\|u\|_r^{\alpha_1}} - C, \end{aligned}$$

that is,

$$\|u\|_r \leq C(\ln(|I_\theta(u)| + 1)^{\frac{1}{\alpha_1}} + C \tag{16}$$

Finally, applying $(f4)$ and (16) we obtain

$$\begin{aligned} |\frac{\partial}{\partial\theta}I(\theta,u)| &= |\int_\Omega \frac{\partial}{\partial\theta}F(\theta,x,u)dx| \\ &\leq C\|u\|_p^r + C \\ &\leq C(\ln(|I_\theta(u)|+1)^{\frac{r}{\alpha_1}} + C, \end{aligned}$$

that is our thesis. □

Remark 3. *Note that hypothesis* $(f2)$ *says, roughly speaking, that the maximal growth allowed for* $F(\cdot,\cdot,\xi)$ *is polynomial, whereas the nonlinear term* $G(\cdot,\cdot,\xi)$ *has an exponential growth: this difference seems not to be removable in our proof, since it is fundamental to obtain the logarithmic estimate of Lemma* 3.4*. Therefore, the arguments presented here seem not to be applicable to problems where the perturbation arises from a nonhomogeneous boundary conditions, such as the following model problem:*

$$\begin{cases} -\Delta u = ue^{|u|^q} & on\ \ \Omega \\ \\ u = u_0 & on\ \ \partial\Omega. \end{cases}$$

Indeed, the functional associated to this problem presents a perturbation term with exponential growth (see for example [5] *for the case* $\mathbb{N} \geq 3$*), which seems too higher to be treated with this method.*

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