Numerical methods for PDEs 1

16.10.2018 – problem set n. 2

Practical part

2.1. **Implementation of a finite element method.** Implement the linear finite element method for the boundary value problem

$$-u'' = f \text{ in } [a,b], \qquad u(a) = 0, \qquad u'(b) = 0,$$

where $a, b \in \mathbb{R}$ with $a < b$. More precisely, implement a function, for example in C of the type

```
void SolveLins(int N, double *mesh, double *sol)
```

where

`N`: stands for the dimension of the discrete space,

`mesh`: points to an array of `double` containing the nodes of the mesh and

`sol`: points to an array of `double` where the coefficients of the approximate solution can be stored.

To this end,

- assume that the load term f is given as a function, for example in C of the type

  ```
  double f(double x)
  ```
- use the approach of problem 2.2 to assemble the linear system by means of a loop over the small intervals, computing the stiffness matrix on the reference interval by hand,
- use the function `thomas` of problem 1.1 to solve the assembled linear system.

Ensure that, apart from the zeros in the stiffness matrix, the linear system is assembled with a number of operations that is of order `N`.

Moreover, implement a function, for example in C of the type

```
double *MakeUnifMesh(double a, double b, int NPoints,
                     double *mesh)
```

that returns the pointer to an array containing the `NPoints` equidistant nodes of a uniform mesh of the interval [`a`,`b`]. If `mesh == NIL`, allocate the memory for the array, otherwise use the address `mesh`.

Test `MakeUnifMesh` and `SolveLins`.

THEORETICAL PART

2.2. **Matrix assembly in 1d.** Consider the linear system for the linear finite element solution of the problem

$$-u'' = f \text{ in } [a,b], \qquad u(a) = 0, \qquad u'(b) = 0,$$

where $a, b \in \mathbb{R}$ with $a < b$. Let K be the matrix of that system, which is refered to as stiffness matrix, and let

$$\hat{K} := \left(\int_0^1 \hat{\phi}'_J \hat{\phi}'_I \right)_{I,J=0,1}$$

with

$$\hat{\phi}_0(x) = 1 - x, \quad \hat{\phi}_1(x) = x, \quad (x \in [0,1]),$$

be the stiffness matrix on the reference interval. Verify that the stiffness matrix K can be computed in the following manner:

$$K = \sum_{l=1}^{n} K^l$$

where the coefficients of the matrices K^l are

$$K^l_{ij} = \begin{cases} h_l^{-1} \hat{K}_{i-l+1,j-l+1} & \text{if } i,j \in \{l-1, l\}, \\ 0, & \text{otherwise,} \end{cases} \qquad (i,j = 1, \ldots, n)$$

with $h_l = x_l - x_{l-1}$. Using this approach, how much memory and how many operations are needed?

Hint: Interpret I, J as local and i, j as global indices, which are related through the number l of the interval.

2.3. **Classical integration by parts with test functions.** Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a nonempty open set, $i \in \{1, \ldots, d\}$ and $v \in C^0(\Omega)$ be such that $\partial_i v \in C^0(\Omega)$. Prove that

$$\forall \varphi \in C^1_0(\Omega) \qquad \int_\Omega \partial_i v \varphi = -\int_\Omega v \partial_i \varphi.$$

Hint: Consider first the case $d = 1$ and use it for the general case, assuming, e.g., $[a,b] \times \mathbb{R}^{d-1} \supseteq \Omega$ for $i = 1$.

2.4. **Derivatives of polynomials.** Given multi-indices $\alpha, \beta \in \mathbb{N}_0^d$, show that

$$\partial^\beta x^\alpha = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} & \text{if } \alpha \geq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

2.5. **Axioms for finite element spaces.** The book 'The Finite Element Method for Elliptic Problems' (North-Holland, Amsterdam, 1978) by P. G. Ciarlet describes a finite element space $V(\mathcal{M})$ over a mesh $\mathcal{M}$ with the following properties:

FEM1: The closure $\overline{\Omega}$ of the domain $\Omega \subset \mathbb{R}^d$ is subdivided in subsets $K \in \mathcal{M}$ such that
- (a) For any $K \in \mathcal{M}$, the set K is closed and its interior is non-empty and connected.
- (b) (if $d \geq 2$) For any $K \in \mathcal{M}$ the boundary ∂K is Lipschitz.
- (c) $\overline{\Omega} = \cup_{K \in \mathcal{M}} K$.
- (d) If $K_1, K_2 \in \mathcal{M}$ are different, then $\operatorname{int}(K_1) \cap \operatorname{int}(K_2) = \emptyset$.

FEM2: Setting $P_K := \{v_{|K} \mid v \in V(\mathcal{M})\}$, the space P_K contains polynomials or functions "close to" polynomials.

FEM3: $V(\mathcal{M})$ has a "computable" basis and the supports of the basis functions are small.

Verify that, for $k \in \{1, 2\}$ and

$$0 = x_0 < x_1 < \cdots < x_i < \cdots < x_n < x_{n+1} = 1,$$

the space

$$S_k = \{v \in C^0[0,1] \mid \forall i = 1, \ldots, n+1 \ v_{|[x_{i-1}, x_i]} \in \mathbb{P}_k[x_{i-1}, x_i], \\ v(0) = 0 = v(1)\}$$

are finite element spaces.

INFORMATION:

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