Numerical methods for PDEs 1

23.10.2018 – problem set n. 3

Practical part

3.1. **Error decay for linear finite elements.** Using the implementation of Problem 2.1, observe the error decay for the linear finite element method. More precisely:

- Implement a function that inserts in a given mesh all midpoints of the subintervals as new points, thus producing a new mesh with maximal subinterval length divided by 2.
- Implement a function that computes an approximation of the error in the energy norm

$$\|u' - U'\|_{0,2;]0,1[} := \left(\int_0^1 |u' - U'|^2 \right)^{1/2},$$

  by means of a loop over the subintervals and by applying Simpson's rule in each subinterval.
- Choose a problem with known solution and, starting from a uniform mesh, observe the so-called EOC (experimental order of convergence) given by

$$EOC_k := \log_2 \frac{e_{k-1}}{e_k}$$

  where k indicates the iteration number and e_k the error in iteration k.

Test your implementations by approximating the solution of

$$-u'' = 1 \text{ in }]0,1[, \quad u(0) = 0 = u(1).$$

Theoretical part

3.2. **Minimal surface equation.** Given an open set $\Omega \subset \mathbb{R}^d$ and $g \in C^0(\partial\Omega)$, define

$$A[v] := \int_\Omega \sqrt{1 + |\nabla v|^2}$$

and

$$V_g := \{ v \in C^1(\overline{\Omega}) \mid v = g \text{ su } \partial\Omega \}.$$

Verify that if u is a minimizer of A in V_g, then

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \text{ in } \Omega, \quad u = g \text{ su } \partial\Omega.$$

Moreover, check that the operator

$$v \mapsto -\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)$$

is nonlinear.

3.3. **Elliptic problems from implicit time discretizations.** Given $T > 0$ and an open set $\Omega \in \mathbb{R}^d$, consider the following problem: find

$$u : \bar{\Omega} \times [0, T] \to \mathbb{R}$$

such that

$$\begin{aligned} \partial_t u - \operatorname{div}(A\nabla u) &= f \quad \text{in } \Omega \times]0, T[\\ \partial_n u &= 0 \quad \text{on } \partial\Omega \times]0, T[\\ u(\cdot, 0) &= v \quad \text{on } \Omega, \end{aligned} \tag{1}$$

where $A(x, t)$ is a symmetric definite positive matrix for all $(x, t) \in \Omega \times]0, T[$. Moreover, assume that

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$$

is a partition of $[0, T]$. Show that the time stepping problems of the implicit Euler method for (1) are (elliptic) boundary value problems. What happens to these problems when the time step tends to 0?

3.4. **A pure Neumann problem.** Consider the following problem: Given

- a domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with C^1 boundary and exterior normal n,
- a matrix function $A \in C^1(\Omega; \mathbb{R}^{d\times d}) \cap C^0(\overline{\Omega}; \mathbb{R}^{d\times d})$ such that $A(x)$ is symmetric and positive definite for all $x \in \overline{\Omega}$,
- a source $f \in C^0(\overline{\Omega})$ and
- boundary values $g \in C^0(\partial\Omega)$,

find $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega, \qquad A\nabla u \cdot n = g \quad \text{on } \partial\Omega.$$

Prove:

(a) If u is a solution and $c \in \mathbb{R}$ is a constant, then $u + c$ is also a solution.
(b) If u_1 and u_2 are solutions, then there exists a constant $c \in \mathbb{R}$ such that $u_1 - u_2 = c$.
(c) If there is a solution, then there holds $\int_\Omega f + \int_{\partial\Omega} g = 0$.

3.5. **Finite elements and finite differences.** Consider the problem

$$-u'' = f \text{ in }]0,1[, \qquad u(0) = 0 = u(1)$$

with $f \in C^0[0,1]$. Given a uniform mesh

$$\mathcal{M}: \qquad x_i = \frac{i}{n} \quad (i = 0, \dots, n \in \mathbb{N}),$$

set

$$S := \big\{ V \in C^0[0,1] \mid \forall i = 1, \dots n \; V_{|]x_{i-1},x_i[} \in \mathbb{P}_1(]x_{i-1}, x_i[), \\ V(0) = 0 = V(1) \big\}$$

and recall the trapezoidal rule

$$\int_0^1 g \approx \frac{1}{2} \sum_{i=1}^n (x_i - x_{i-1}) \big(g(x_{i-1}) + g(x_i) \big)$$

Let $U_{\mathrm{FE}} \in S$ be such that

$$\forall \varphi \in S \quad \int_0^1 U'_{\mathrm{FE}} \varphi' = \frac{1}{2} \sum_{i=1}^n (x_i - x_{i-1}) \big((f\varphi)(x_{i-1}) + (f\varphi)(x_i) \big)$$

and $(U_{\mathrm{FD},i})_i$ be such that

$$\frac{-U_{\mathrm{FD},i-1} + 2U_{\mathrm{FD},i} - U_{\mathrm{FD},i+1}}{h^2} = f(x_i) \quad i = 1, \dots n-1$$
$$U_{\mathrm{FD},0} = 0 = U_{\mathrm{FD},n}$$

with $h = 1/n$. Verify that

$$\forall i = 0, \dots, n \qquad U_{\mathrm{FE}}(x_i) = U_{\mathrm{FD},i}.$$

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