Numerical methods for PDEs 1

06.11.2018 – problem set n. 5

Practical part

5.1. **Simulation of pressure sensor.** Solve with ALBERTA the problem

$$-\Delta u = p_e - p_i \quad \text{in} \quad]-1,1[^2, \qquad u = 0 \quad \text{on} \quad \partial]-1,1[^2,$$

where the constants p_e, p_i denote, respectively, the pressure exterior and interior to the sensor. To this end:

- Copy the files `ellipt.c`, `graphic.c` of the folder `Common/` and the files `Makefile`, `INIT/ellipt.dat`, `Macro/macro.amc` of the folder `2d/` into a new folder of your choice.
- Modify the copy of `ellipt.c`, implementing load term and boundary values.
- Compile the source codes after suitably modifying `make`.
- Run and visualize the numerical solution for various pressures, choosing the following values in the parameter file `ellipt.dat`:

```
macro file name:          Macro/macro.amc
polynomial degree:        1
adapt->strategy:          1
adapt->max_iteration:     5
```

  Accordingly, starting from the mesh in `Macro/macro.amc`, the program performs a global refinement and computes the corresponding approximations with linear finite elements for five times.

Theoretical part

5.2. **Bilinear forms and linear operators.** Let V and W be linear spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively, and write $V' := \mathrm{Lin}(V,\mathbb{R})$ and $W' := \mathrm{Lin}(W,\mathbb{R})$ for their (topological) dual spaces. A bilinear form $b : V \times W \to \mathbb{R}$ is bounded whenever there exists a constant $C \geq 0$ such that

$$\forall v \in V, w \in W \qquad |b(v,w)| \leq C \, \|v\|_V \, \|w\|_W \, .$$

Show that, by

$$\forall v \in V, w \in W \quad \langle Lv, w\rangle = b(v,w),$$

every bounded bilinear form $b : V \times W \to \mathbb{R}$ defines a linear operator $L \in \mathrm{Lin}(V, W')$ and vice versa.

5.3. **Completeness and examples.** A linear space V with norm $\|\cdot\|_V$ is complete if every Cauchy sequence has a limit in V, i.e. if $(v_k)_k$ satisfies $\|v_m - v_n\| \to 0$ as $n, m \to \infty$, then there exists $v \in V$ such that $\|v_k - v\| \to 0$ as $k \to \infty$. In this case, one says also that $(V, \|\cdot\|_V)$ is a Banach space.

Let $\Omega \subset \mathbb{R}^d$ be non-empty, open and bounded. Show:

(a) $(C^0(\bar{\Omega}), \|\cdot\|_{0,p;\bar{\Omega}})$ with $p \in [1, \infty[$ is not a Banach space.
(b) $(C^0(\bar{\Omega}), \|\cdot\|_{0,\infty;\bar{\Omega}})$ is a Banach space.
(c) $(C^1(\bar{\Omega}), \|\cdot\|_{0,\infty;\bar{\Omega}})$ is not a Banach space.

5.4. **Extension of linear operators.** Let V and W be linear spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively, and let $\tilde{V}$ be a linear subspace of V. Prove: If $\tilde{V}$ is dense in V and W is complete, then every linear operator $\tilde{L} \in \mathrm{Lin}(\tilde{V}, W)$ has a unique extension $L \in \mathrm{Lin}(V, W)$ such that $L_{|\tilde{V}} = \tilde{L}$.

5.5. **Lebesgue norms.** Let $\Omega \subset \mathbb{R}^d$ be a non-empty, open and bounded set and $p \in [1, \infty]$. Set (whenever defined)

$$\|v\|_{0,p;\Omega} := \left(\int_\Omega |v|^p \right)^{1/p} \quad \text{for} \quad p < \infty,$$
$$\|v\|_{0,\infty;\Omega} = \inf_{|N|=0} \sup_{x \in \Omega \setminus N} |v(x)|,$$

where $\int$ denotes the Riemann or the Lebesgue integral and $|\cdot|$ is the Jordan or Lebesgue measure. In the case of Lebesgue integral and Lebesgue measure,

$$L^p(\Omega) := \{v : \Omega \to \mathbb{R} \mid \|v\|_{0,p;\Omega} \text{ is defined and finite}\}$$

is a Lebesgue space associated with $p \in [1, \infty]$. Prove:

(a) If $v \in C^0(\bar{\Omega})$, then $\sup_{x \in \Omega} |v(x)| = \inf_{|N|=0} \sup_{x \in \Omega \setminus N} |v(x)|$.
(b) If $\Omega = B_d := \{x \in \mathbb{R}^d \mid |x| \leq 1\}$ and $\rho \in \mathbb{R}$, then

$$|\cdot|^\rho \in L^p(\Omega) \iff \rho > -\frac{d}{p}.$$

(c) If $\varphi \in C_0^\infty(B_d)$, $\delta > 0$ and $\varphi_\delta := \varphi(\cdot/\delta)$, then

$$\|\varphi_\delta\|_{0,p;\delta B_d} = \delta^{\frac{d}{p}} \|\varphi\|_{0,p;B_d}.$$

(d) $\|\cdot\|_{0,p;\Omega}$ is a norm provided one considers suitable equivalence classes in $L^p(\Omega)$.
(e) If $0 < q \leq p \leq \infty$, then $L^p(\Omega) \subset L^q(\Omega)$ and the linear operator $L^q(\Omega) \ni v \mapsto v \in L^p(\Omega)$ has norm $|\Omega|^{\frac{1}{q} - \frac{1}{p}}$.

Hint: Use polar coordinates for (b) and the Hölder inequality for (d).