Numerical methods for PDEs 1

27.11.2018 – problem set n. 8

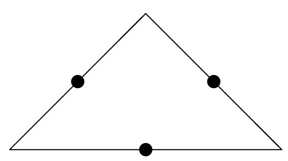
PRACTICAL PART

8.1. **Numerical Integration.** Let $\mathcal{M}$ be a triangulation of a domain $\Omega \subset \mathbb{R}^2$, let $f \in C^0(\bar{\Omega})$ and assume that a quadrature rule of ALBERTA is given. Implement a program that computes approximate values of the integrals $\{\int_K f\}_{K \in \mathcal{M}}$ and $\int_\Omega f$. Test your implementation

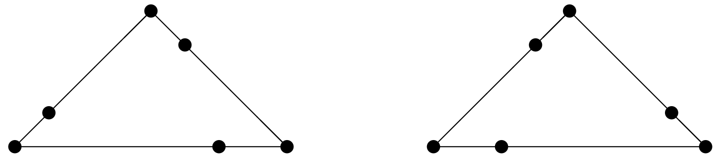
- by considering a case where the exact values of $\{\int_K f\}_{K \in \mathcal{M}}$ are known,
- by observing the experimental order of convergence (EOC) of the error in approximating $\int_\Omega f$ for global mesh refinements.

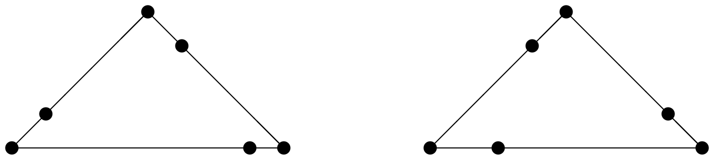
THEORETICAL PART

8.2. **Nodal basis for Crouzeix-Raviart element.** Determine the nodal basis for the so-called Crouzeix-Raviart element, which has the following nodal variables for $\mathbb{P}_1$:

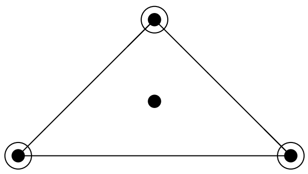


8.3. **Affine equivalent or not?** Decide if the following nodal variables for $\mathbb{P}_2$ define couples of affine equivalent finite elements:

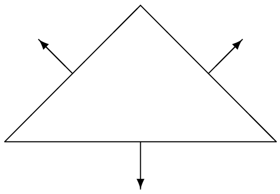




Moreover, consider the nodal variables



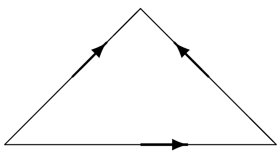
where the big circle indicates the evaluation of the two partial derivatives at the corresponding point for $\mathbb{P}_3$. Is this a finite element? If yes, does the affine equivalence between triangles entail the affine equivalence of this type of element?
Furthermore, consider



where the arrows indicate the evaluation of the normal component at the corresponding points for the space

$$P = \left\{ p : \mathbb{R}^2 \to \mathbb{R}^2 \mid \exists c_1, c_2, c \in \mathbb{R} \; \forall x \in \mathbb{R}^2 \; p(x) = c_1 e_1 + c_2 e_2 + cx \right\}$$

of vector-valued functions. Check that this is a finite element. Does the affine equivalence between triangles entail the affine equivalence of this type of element?
Finally, consider



where the arrows indicate the evaluation of the tangential component at the corresponding points for the space

$$P = \Big\{p : \mathbb{R}^2 \to \mathbb{R}^2 \mid \exists c_1, c_2, c \in \mathbb{R}\ \forall x \in \mathbb{R}^2\ p(x) = c_1 e_1 + c_2 e_2 + c \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\Big\}$$

of vector-valued functions. Check again that this is a finite element and clarify if the affine equivalence between triangles entails the affine equivalence of this type of element?

8.4. **Properties of Lagrange lattice.** Let $L_\ell(K)$ be the standard Lagrange lattice of order $\ell \in \mathbb{N}$ of a n-simplex K in $\mathbb{R}^d$. Prove the following properties of $L_\ell(K)$: for any subsimplex S of K and any affine injection $F : K \to \mathbb{R}^m$, $m \in \mathbb{N}$, we have

$$\# L_\ell(K) = \dim \mathbb{P}_\ell(K), \qquad L_\ell(K \cap S) = L_\ell(K) \cap S,$$
$$F\big(L_\ell(K)\big) = L_\ell\big(F(K)\big).$$

8.5. **Graphs of Lagrange basis functions.** Consider the standard Lagrange basis functions of order $\ell \in \{1, 2, 3\}$ over a triangulation in $\mathbb{R}^2$. Sketch their graphs for interior and boundary nodes.

Information:

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