Numerical methods for PDEs 1

04.12.2018 – problem set n. 9

PRACTICAL PART

9.1. **Lagrange interpolation.** Let u be a continuous function on a planar domain $\Omega \subset \mathbb{R}^2$. Implement with the help of ALBERTA a program that plots the graph of the Lagrange interpolation $u_{\ell,\mathcal{M}}$ of u given by

$$\forall K \in \mathcal{M} \qquad u_{\ell,\mathcal{M}|K} \in \mathbb{P}_\ell \quad \text{and} \quad \forall z \in L_\ell(K) \;\; u_{\ell,\mathcal{M}}(z) = u(z),$$

where $\mathcal{M}$ is an edge-to-edge triangulation of Ω and $L_\ell(K)$ denotes the principal Lagrange lattice of the simplex K.

For this purpose:

- Compute the vector of type `DOF_REAL_VEC` that contains the coefficients identifying $u_{\ell,\mathcal{M}}$ in the following manner: execute a non-recursive mesh traversal and, on any leaf, use the function

  `fe_space->bas_fcts->interpol()`,

  where `FE_SPACE *fe_space` denotes the global variable representing the finite element space $S^{\ell,0}(\mathcal{M})$.
- Plot with the function `graphics()`.

Consider the partition $\mathcal{M}$ of $\Omega =]0,1[^2$ that is obtained by two refinements of any triangle in the macro triangulation which in turn is given by the two diagonals of Ω. Plot the Lagrange interpolation $u_{1,\mathcal{M}}$ of the function

$$u(x_1, x_2) = 1 - (x_1 - 0.4)^2 - (x_2 - 0.7)^2, \quad x = (x_1, x_2) \in]0,1[^2.$$

Moreover, using the macro `FOR_ALL_DOFS` compute the largest coefficient of $u_{1,\mathcal{M}}$. Is this the maximum $u_{1,\mathcal{M}}$ in Ω? Justify your answer.

THEORETICAL PART

9.2. **Boundary and nodal values.** Let $\mathcal{M}$ be a simplicial mesh of a domain $\Omega \subset \mathbb{R}^d$, $\mathcal{P} = \{\mathbb{P}_\ell(K)\}_{K\in\mathcal{M}}$ with $\ell \in \mathbb{N}$ and write $\mathcal{N}$ for the Lagrange nodes of $\mathcal{M}$. Verify the identity

$$\Big\{V \in \mathrm{FE}(\mathcal{M}, \mathcal{P}, C^0(\overline{\Omega})) \mid V_{|\partial\Omega} = 0\Big\} = \Big\{V \in \mathrm{FE}(\mathcal{M}, \mathcal{P}, C^0(\overline{\Omega})) \mid \forall z \in \mathcal{N} \cap \partial\Omega \;\; V(z) = 0\Big\}.$$

Why is it useful?

9.3. **Basis for continuous piecewise bilinear functions.** Let $\mathcal{M}$ be a mesh of a domain $\Omega \subset \mathbb{R}^2$ into rectangles, i.e. into sets of the type $R = [a_1, b_1] \times [a_2, b_2]$ with $a_1 < b_1$ and $a_2 < b_2$. Moreover, denote the set of vertices of $\mathcal{M}$ by $\mathcal{V}$. Under suitable assumptions on $\mathcal{M}$, construct a basis of the type

$$\{\Phi_z\}_{z \in \mathcal{V}} \quad \text{such that} \quad \forall y \in \mathcal{V} \;\; \Phi_z(y) = \delta_{yz}$$

of the space

$$\{v \in C^0(\overline{\Omega}) \mid \forall R \in \mathcal{M} \; v_{|R} \in Q_1(R)\}$$

where

$$Q_1(R) = \{q : R \to \mathbb{R} \mid \forall x = (x_1, x_2) \in R \qquad q(x) = c_{00} + c_{10}x_1 + c_{01}x_2 + c_{11}x_1x_2\}.$$

9.4. **Point values and weak derivatives.** Let $d \geq 2$. Show that the function

$$v(x) := \ln|\ln|x||, \quad x \in \Omega := \{x \in \mathbb{R}^d \mid |x| < \tfrac{1}{2}\}$$

verifies $v \in W^{1,p}(\Omega) \setminus L^\infty(\Omega)$ whenever $p \in [1, d]$.

Hint: Write

$$\int_\Omega v \partial_i \varphi = \lim_{\epsilon \to 0} \int_{\Omega \setminus B_\epsilon(0)} v \partial_i \varphi$$

before integrating by parts.

9.5. **First order Friedrichs inequality.** Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ be non-empty, open and bounded. Prove that, for all $v \in W_0^{s,p}(\Omega)$, we have

$$\|v\|_{0,p;\Omega} \leq \operatorname{diam}(\Omega) \, |v|_{1,p;\Omega} \, .$$

Hint: Prove the statement first for $v \in C_0^\infty(\Omega)$ and $d = 1$. Then exploit $\Omega \subset [a, b]^d$ for suitable $a, b \in \mathbb{R}$.

9.6. **Weak Laplace operator.** Let $\Omega \subset \mathbb{R}^d$ be a domain and endow $H_0^1(\Omega)$ with the norm

$$|\cdot|_{1,2;\Omega} = \| \, |\nabla \cdot| \, \|_{0,2;\Omega},$$

which arises from the scalar product

$$H_0^1(\Omega) \ni (v, w) \mapsto \int_\Omega \nabla v \cdot \nabla w.$$

Given $v \in H_0^1(\Omega)$, define the (weak) Laplace of v by

$$\langle -\Delta v, \varphi \rangle := \int_\Omega \nabla v \cdot \nabla \varphi, \qquad \varphi \in H_0^1(\Omega).$$

Prove:

(a) $-\Delta$ is a linear bounded operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.
(b) The inverse of $-\Delta$ is the Riesz representation isometry of $H_0^1(\Omega)$.