Numerical methods for PDEs 1

11.12.2018 – problem set n. 10

Practical part

10.1. **Distribution and EOC of interpolation error.** Implement with the help of ALBERTA a program that plots the distribution over elements of the Lagrange interpolation errors

$$||u - u_{\ell;\mathcal{M}}||_{0;\Omega} \quad \text{and} \quad |u - u_{\ell;\mathcal{M}}|_{1;\mathcal{M}}$$

and computes their EOCs.

In order to plot the error distribution and to prepare the computation of the EOCs, proceed as follows:

- Use the library functions `L2_err()` and `H1_err()`, after having defined an appropriate structure to store the square of the local errors on each leaf element of the hierarchical mesh and after having initialized the data structure `LEAF_DATA_INFO` correspondingly.
- Plot the local errors in a graphic window by calling

  ```
  graphics(mesh, nil, get_el_err);
  ```

  where `get_el_err()` is a user defined function which returns the error values stored at each leaf element of the mesh.

Consider

$$u_1(x_1, x_2) = x_1^3 + x_2^3 \quad \text{and} \quad u_2(x) = \sin(2\pi x_1)$$

for $x = (x_1, x_2) \in \,]0, 1[^2$ with uniform refinements of the macro triangulation in Problem 9.1 for $\ell = 1, 2, 3$.

10.2. **Assembly of stiffness matrix.** Given a domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, and the bilinear form

$$b(v, \varphi) := \int_\Omega \nabla v \cdot \nabla \varphi, \qquad v, \varphi \in H^1(\Omega),$$

carry out the following tasks:

- Implement the assembly of the stiffness matrix

  $$K_{ij} := b(\Phi_j, \Phi_i), \qquad i, j = 1, \dots, N$$

  where $(\Phi_i)_{i=1}^N$ is the nodal basis of $S^{\ell,0}(\mathcal{M})$ with $\ell \in \{1, 2, 3\}$ over a simplicial partition of Ω with matching faces.
- Verify and implement in matrix-vector form the identity

$$(*) \qquad \forall i = 1, \dots, N \quad \sum_{j=1}^N K_{ij} = 0.$$

- Numerically verify $(*)$ for the available macro triangulations of $\Omega =]0,1[^2$ and $\Omega =]-1,1[^2 \setminus (]0,1[\times]-1,0[)$.

Theoretical part

10.3. **Inclusion $L^2 \subset H^{-1}$.** Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a domain and $1 < p < \infty$. Verify that $L^p(\Omega) \subset W^{-1,p}(\Omega)$ with

$$\forall f \in L^p(\Omega) \qquad \|f\|_{-1,p;\Omega} \leq \operatorname{diam}(\Omega)\|f\|_{0,p;\Omega}.$$

10.4. **Reaction-diffusion problem.** Consider

$$-\Delta u + ru = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where the coefficient $r = r(x)$ is positive. Propose a notion of weak solution and verify its well-posedness under suitable assumptions on the data. Moreover, derive a bound for the condition number of the associated bilinear form.

10.5. **Céa Lemma.** In 1968 Birkhoff, Schultz and Varga have individuated the following statement, which is named after Jean Céa who has proved the symmetric case in 1964.

Let V be a Hilbert space with norm $\|\cdot\|$, $\ell \in V^*$ and b a continuous and coercive bilinear form, that is, there exist $M \geq \alpha > 0$ such that

$$\forall v, \varphi \in V \quad |b(v,\varphi)| \leq M\|v\|\|\varphi\|,$$
$$\forall v \in V \qquad b(v,v) \geq \alpha\|v\|^2.$$

Moreover let S be a subspace of V. Verify the following statement: if $u \in V$, $U \in S$ are such that

$$\forall \varphi \in V \quad b(u,\varphi) = \ell(\varphi)$$

and

$$\forall \varphi \in S \quad b(U,\varphi) = \ell(\varphi),$$

then there holds

$$\|u - U\| \leq \frac{M}{\alpha} \inf_{v \in S} \|u - v\|.$$

10.6. **Lagrange interpolation in the maximum norm.** Given

$$0 = x_0 < x_1 < \cdots < x_i < \cdots < x_n < x_{n+1} = 1$$

and $v \in C^0[0,1]$, define Iv by

$$Iv(x) := \frac{x_i - x}{x_i - x_{i-1}} v(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} v(x_i) \qquad (x \in [x_{i-1}, x_i],\ i = 1, \dots, n+1),$$

the meshsize function $h \in L^\infty(]0,1[)$ by

$$h_{|[x_{i-1},x_i]} = x_i - x_{i-1},$$

and show that:

(a) if $v \in C^1[0,1]$ (or v Lipschitz), then

$$\sup_{]0,1[} |v - Iv| \leq 2 \sup_{]0,1[} |hv'|$$

(b) if $v \in C^2[0,1]$, then

$$\sup_{]0,1[} |v' - (Iv)'| \leq \sup_{]0,1[} |hv''|$$

and

$$\sup_{]0,1[} |v - Iv| \leq \sup_{]0,1[} |h^2 v''|.$$

Compare these three error bounds.

10.7. **Polynomial for Bramble-Hilbert lemma.** Given $\ell \in \mathbb{N}_0$, a domain $\Omega \subset \mathbb{R}^d$ and $v \in H^\ell(\Omega)$, verify that there exists a unique polynomial $P \in \mathbb{P}_\ell(\Omega)$ such that

$$\forall |\alpha| \leq \ell \quad \int_\Omega \partial^\alpha P = \int_\Omega \partial^\alpha v.$$

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