Numerical methods for PDEs 1

18.12.2018 – problem set n. 11

PRACTICAL PART

11.1. **Dirichlet boundary condition and solution of linear system.** Complete the implementation of the program that numerically solves the problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega,$$

by including the Dirichlet boundary condition in the linear system and solving the resulting linear system.

In order to verify the implementation, consider the domain $\Omega =]0,1[^2$ with the usual macro triangulation, f and g such that

$$\begin{aligned} u_1(x_1,x_2) &= 0, \\ u_2(x_1,x_2) &= 1, \\ u_3(x_1,x_1) &= x_1(1-x_1)x_2(1-x_2), \\ u_4(x_1,x_2) &= x_1^3 \end{aligned}$$

are exact solutions and refine globally (if useful).

Moreover, compute approximations to the solutions u_5 and u_6 for the data

$$f_5 = \chi_{]1/4,3/4[^2}, \quad g_5 = 0;$$
$$f_6 = 0, \quad g_6(x_1,x_2) = (x_1 - 1/4)^2 + (x_2 - 3/4)^2$$

and give examples for physical situations that could have been modeled.

THEORETICAL PART

11.2. **Scalings of nodal basis and variables.** Let K be a n-simplex in $\mathbb{R}^d$ with $n \le d$ and ϕ_z, $z \in L_\ell(K)$, the nodal basis of the Lagrange element of order ℓ on K. Verify:

(a) There exists a constant C_0 depending on n and ℓ such that

$$\forall z \in L_\ell(K) \quad \|\phi_z\|_{0,2;K} \le C_0 |K|^{1/2}.$$

(b) If $n = d-1$ and ϕ_z^*, $z \in L_\ell(K)$, is the L^2-dual basis of the preceding problem, there exists a constant C_0^* depending on d and ℓ such that

$$\forall z \in L_\ell(K) \quad \|\phi_z^*\|_{0,2;K} \le C_0^* |K|^{-1/2}.$$

(c) If $n = d$, there exists a constant C_1 depending on d and ℓ such that

$$\forall z \in L_\ell(K) \quad \|\nabla \phi_z\|_{0,2;K} \le C_1 \frac{|K|^{1/2}}{\rho_K}.$$

11.3. **Two more settings for well-posedness.** Let $\Omega \subset \mathbb{R}^d$ be a domain and consider the problem

$$\text{find } u \in H_0^1(\Omega) \text{ such that } \forall \varphi \in H_0^1(\Omega) \quad \int_\Omega A\nabla u \cdot \nabla \varphi = \langle f, \varphi \rangle$$

where $f \in H^{-1}(\Omega)$ and $A(x)$, $x \in \Omega$, is symmetric and there exist constants $\Lambda \ge \lambda > 0$ such that

$$\forall \xi \in \mathbb{R}^d \quad \lambda|\xi|^2 \le A\xi \cdot \xi \le \Lambda|\xi|^2.$$

Compare the two settings in which $H_0^1(\Omega)$ is equipped with $\|\cdot\|_{1,2;\Omega}$ or with the so-called energy norm

$$\|v\| := \left(\int_\Omega A\nabla v \cdot \nabla v \right)^{1/2}.$$

Hint: In particular, investigate the continuity and coercivity of the bilinear form and the use of the Friedrichs inequality.

11.4. **Convergence and global refinement.** Verify with an example that global refinement is in general not necessary for the convergence of the Galerkin approximations associated to mesh sequence.

Hint: Use and verify that Lagrange interpolation in 1d yields the linear finite element solution for $u'' = f$.