Exam problems of **Numerical methods for PDEs 1**

year 2018/19 – valid until January 2020

1. **One-dimensional convection-diffusion problems.** Implement, in the programming language of your choice, a program to approximate one-dimensional boundary value problems of the form

$$-u'' + bu' = 0 \text{ on }]0,1[, \quad u(0) = 1,\ u(1) = 0$$

with the help of linear and quadratic elements.

Determine the exact solution and apply your implementation, with global refinements reaching at least a maximum meshsize of 10^{-3}, to approximate it for $b = 0.5, 5, 50, 500$. Observe and comment the approximate solutions as well as their error decay in the (appropriately approximated) H^1-error.

2. **Investigating the quasi-optimality constant.** Modify the source code `ellipt.c` in order to solve a problem of the form

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega,$$

where A is constant diagonal matrix, and to compute the so-called Riesz projection U^* into $S^{\ell,0}_g(\mathcal{M}) := \{v \in S^{\ell,0}(\mathcal{M}) \mid v = I_{\mathcal{M}} g\}$ given by

$$\forall \varphi \in S^{\ell,0}_0(\mathcal{M}) \quad \int_\Omega \nabla U^* \cdot \nabla\varphi = \int_\Omega \nabla u \cdot \nabla \varphi$$

and $I_{\mathcal{M}}$ denotes the Lagrange interpolation of order ℓ on the mesh $\mathcal{M}$.

Apply your implementation, with Lagrange elements of order $\ell \in \{1, 2, 3\}$ and global refinements, in order to approximate the exact solutions

$$u_1(x_1, x_2) = \sin(\pi x_1), \quad u_2(x_1, x_2) = \sin(\pi x_2),$$
$$u_3(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$$

with

$$\Omega =]0,1[^2, \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha \in \{1, 10, 100\}.$$

Observe, compare, and comment the errors of the Galerkin approximation and the Riesz approximation in the H^1-seminorm.

3. **Approximation of output functionals.** Consider the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

and assume that we are interested in $\ell(u)$, where $\ell \in H^{-1}(\Omega)$ is a so-called output functional.

Implement, e.g. within ALBERTA, a program that computes $\ell(U)$ as an approximation for $\ell(u)$, assuming that ℓ is given as C function.

Use your implementation to determine the experimental order of convergence of $|\ell(U) - \ell(u)|$ in the following cases:

$$\Omega_1 = (-1,1)^2, \quad u_1(x_1,x_2) = (x_1+1)(x_1-1)(x_2+1)(x_2-1),$$

$$\ell_1(v) = \tfrac{1}{4}\int_{\Omega_1} v,$$

$$\Omega_2 = \Omega_1 \setminus \big((0,1)\times(-1,0)\big), \quad u_2(r,\varphi) = r^{2/3}\sin\frac{2\varphi}{3},$$

$$\ell_2 = \tfrac{1}{3}\int_{\Omega_2} v,$$

$$\Omega_3 = \Omega_1, \quad u_3 = u_1, \quad \ell_3(v) = v(0).$$

Compare and interpret the differences with the behavior of the errors $\|\nabla(u-U)\|_{0,2;\Omega}$, $\|u-U\|_{0,2;\Omega}$ as well as the available theoretical results.

INFORMATION:

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