## TEMA A

**ESERCIZIO 2**

$$K(f(x)) = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x\ e^{2x^2}\ 4x}{e^{2x^2}}\right| = 4x^2$$

$$4x^2 < 100 \iff -5 < x < 5$$

**ESERCIZIO 3**

$\alpha = 2$
1) $f(1) = 1 - 8 = -7 < 0,\ f(4) = 64 - 8 = 56 > 0;$
2) $f'(x) = 3x^2 > 0,\ \forall x \in [1,4];\ f'(1) = 3,\ f'(4) = 48;$
3) $f''(x) = 6x > 0,\ \forall x \in [1,4];$
4) $\left|\frac{f(1)}{f'(1)}\right| = \frac{7}{3} < 3;\ \left|\frac{f(4)}{f'(4)}\right| = \frac{56}{48} < 3;$
1)+2)+3)+4) $\Rightarrow$ Il metodo di Newton converge ad $\alpha\ \forall x_0 \in [1,4]$

$$x_{k+1} = x_k - \frac{x_k^3 - 8}{3x_k^2} = \frac{2x_k^3 + 8}{3x_k^2} = \frac{2}{3}x_k + \frac{8}{3x_k^2}$$

$$g(x) = \frac{2}{3}x + \frac{8}{3x^2},\quad g(2) = 2$$

Studio di $y = g(x)$.
C.E. $x \neq 0$, asintoto verticale $x = 0$, asintoto obliquo $y = \frac{2}{3}x$.
$g(x) > 0$ se $2x^3 + 8 > 0$, cioè $x > -\sqrt[3]{4}$
$g'(x) = \frac{2}{3} - \frac{16}{3x^3} = \frac{2x^3 - 16}{3x^3} > 0$, se $x < 0\ \vee x > 2;$

Minimo: $(2, 2)$

Studio della convergenza al variare di $x_0$
$0 < x_0 < \alpha \longrightarrow x_1 > \alpha;$
$x_0 > \alpha \longrightarrow x_n \searrow \alpha$
(successione monotona decrescente limitata inferiormente da $\alpha$)

$g'(2) = 0$ (ovvio per il metodo di Newton, $\alpha = 2$ ha molteplicità 1)
$g''(2) \neq 0 \Longrightarrow 2^o$ ordine.

(Facoltativo) Condizione sufficiente per la convergenza:

$$|g'(x)| < 1 \Longleftrightarrow \left|\frac{2x^3 - 16}{3x^3}\right| < 1 \Longleftrightarrow x > \sqrt[3]{\frac{16}{5}}$$

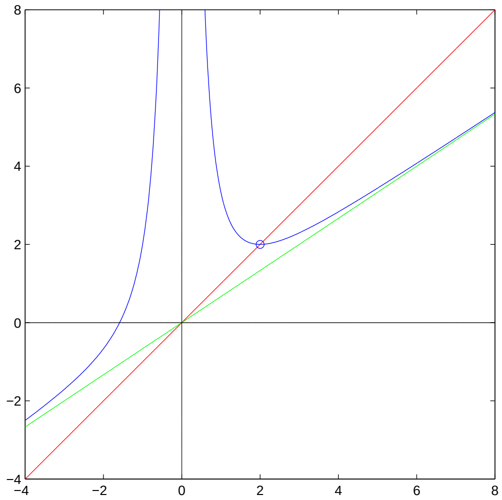


Figure 1: Grafico della funzione $g(x)$ (Tema A)

**ESERCIZIO 4** (Ricordare l'ipotesi $\alpha > 1$)

4.1) Calcolo della fattorizzazione $LU$

$$m_{31} = \frac{1}{\alpha^2}, \quad a_{33} = \alpha - \frac{1}{\alpha^2}\frac{1}{\alpha} = \alpha - \frac{1}{\alpha^3}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{\alpha^2} & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \alpha & 0 & \frac{1}{\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & \alpha - \frac{1}{\alpha^3} \end{pmatrix}}_{U}$$

$$\|A\|_\infty = \alpha + \frac{1}{\alpha}$$

$$\|L\|_\infty = 1 + \frac{1}{\alpha^2}, \quad \|U\|_\infty = \max\left\{\alpha + \frac{1}{\alpha}, \alpha - \frac{1}{\alpha^3}\right\} = \alpha + \frac{1}{\alpha}$$

$$(\|A\|_\infty =)\alpha + \frac{1}{\alpha} < (\|L\|_\infty \|U\|_\infty =)\left(1 + \frac{1}{\alpha^2}\right)\left(\alpha + \frac{1}{\alpha}\right) = \alpha + \frac{1}{\alpha} + \underbrace{\left(\frac{1}{\alpha} + \frac{1}{\alpha^3}\right)}_{>0}$$

4.2) Per dimostrare che il metodo di Jacobi è convergente è sufficiente osservare che la matrice $A$ è diagonalmente dominante. Oppure:

$$\det\begin{pmatrix} \alpha\lambda & 0 & \frac{1}{\alpha} \\ 0 & \lambda & 0 \\ \frac{1}{\alpha} & 0 & \alpha\lambda \end{pmatrix} = \lambda\left(\alpha^2\lambda^2 - \frac{1}{\alpha^2}\right) = 0 \iff \lambda_1 = 0,\ \lambda_{2,3} = \pm\frac{1}{\alpha^2}$$

$$\rho(B_J) = \frac{1}{\alpha^2} < 1 \Longleftrightarrow |\alpha| > 1$$

4.3)

$$A = M - N, \quad N = M - A = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} \\ 0 & \alpha - 1 & 0 \\ -\frac{1}{\alpha} & 0 & 0 \end{pmatrix}$$

$$P = M^{-1}N = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} \\ 0 & \alpha - 1 & 0 \\ -\frac{1}{\alpha} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha^2} \\ 0 & \frac{\alpha-1}{\alpha} & 0 \\ -\frac{1}{\alpha^2} & 0 & 0 \end{pmatrix}$$

$$\|P\|_\infty = \max\Big\{ \underbrace{\frac{\alpha - 1}{\alpha}}_{<1}, \underbrace{\frac{1}{\alpha^2}}_{<1} \Big\} < 1$$

Oppure:

$$\det(P - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & -\frac{1}{\alpha^2} \\ 0 & \frac{\alpha-1}{\alpha} - \lambda & 0 \\ -\frac{1}{\alpha^2} & 0 & -\lambda \end{pmatrix} = 0 \Longleftrightarrow \left(\lambda - \frac{\alpha - 1}{\alpha}\right)\left(\lambda^2 - \frac{1}{\alpha^4}\right) = 0$$

$$\lambda_1 = \frac{\alpha - 1}{\alpha}, \ \lambda_{2,3} = \pm\frac{1}{\alpha^2}, \ \text{con } |\lambda_i| < 1, \ \forall i = 1, 2, 3$$

**TEMA B**

**ESERCIZIO 2**

$$K(f(x)) = \Big|\frac{xf'(x)}{f(x)}\Big| = \Big|\frac{x\ 2e^{x^2}\ 2x}{2e^{x^2}}\Big| = 2x^2$$

$$2x^2 < 50 \iff -5 < x < 5$$

**ESERCIZIO 3**

$\alpha = 3$
1) $f(2) = 8 - 27 = -19 < 0,\ f(4) = 64 - 27 = 37 > 0;$
2) $f'(x) = 3x^2 > 0,\ \forall x \in [2,4];\ f'(2) = 12,\ f'(4) = 48;$
3) $f''(x) = 6x > 0,\ \forall x \in [2,4];$
4) $\left|\frac{f(2)}{f'(2)}\right| = \frac{19}{12} < 2;\ \left|\frac{f(4)}{f'(4)}\right| = \frac{37}{48} < 2;$
1)+2)+3)+4) $\Rightarrow$ Il metodo di Newton converge ad $\alpha$ $\forall x_0 \in [2,4]$

$$x_{k+1} = x_k - \frac{x_k^3 - 27}{3x_k^2} = \frac{2x_k^3 + 27}{3x_k^2} = \frac{2}{3}x_k + \frac{9}{x_k^2}$$

$$g(x) = \frac{2}{3}x + \frac{9}{x^2},\quad g(3) = 3$$

Studio di $y = g(x)$.
C.E. $x \neq 0$, asintoto verticale $x = 0$, asintoto obliquo $y = \frac{2}{3}x$.
$g(x) > 0$ se $2x^3 + 27 > 0$, cioè $x > -\sqrt[3]{\frac{27}{2}}$
$g'(x) = \frac{2}{3} - \frac{18}{x^3} = \frac{2x^3-54}{3x^3} > 0$, se $x < 0\ \vee x > 3$;
Minimo: $(3,3)$

Studio della convergenza al variare di $x_0$
$0 < x_0 < \alpha \longrightarrow x_1 > \alpha;$
$x_0 > \alpha \longrightarrow x_n \searrow \alpha$
(successione monotona decrescente limitata inferiormente da $\alpha$)

$g'(3) = 0$ (ovvio per il metodo di Newton, $\alpha = 3$ ha molteplicità 1)
$g''(3) \neq 0 \Longrightarrow 2^o$ ordine.
(Facoltativo) Condizione sufficiente per la convergenza:

$$|g'(x)| < 1 \iff \Big|\frac{2x^3 - 54}{3x^3}\Big| < 1 \iff x > \sqrt[3]{\frac{54}{5}}$$

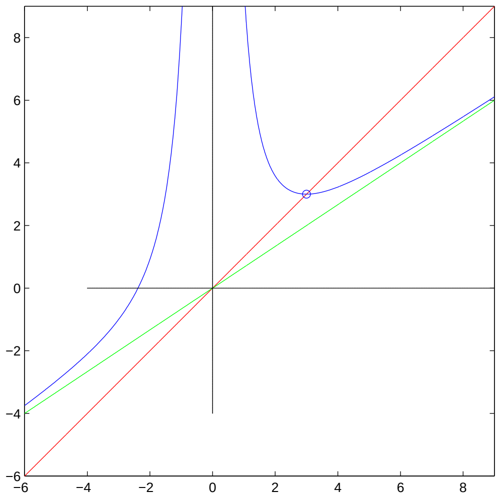


Figure 2: Grafico della funzione $g(x)$ (Tema B)

**ESERCIZIO 4** (Ricordare l'ipotesi $\alpha > 2$)

4.1) Calcolo della fattorizzazione $LU$

$$m_{31} = \frac{4}{\alpha^2}, \quad a_{33} = \alpha - \frac{4}{\alpha^2}\frac{4}{\alpha} = \alpha - \frac{16}{\alpha^3}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{4}{\alpha^2} & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \alpha & 0 & \frac{4}{\alpha} \\ 0 & 4 & 0 \\ 0 & 0 & \alpha - \frac{16}{\alpha^3} \end{pmatrix}}_{U}$$

$$\|A\|_\infty = \alpha + \frac{4}{\alpha}$$

$$\|L\|_\infty = 1 + \frac{4}{\alpha^2}, \quad \|U\|_\infty = \max\left\{\alpha + \frac{4}{\alpha}, \alpha - \frac{16}{\alpha^3}\right\} = \alpha + \frac{4}{\alpha}$$

$$(\|A\|_\infty =)\alpha + \frac{4}{\alpha} < (\|L\|_\infty \, \|U\|_\infty =) \left(1 + \frac{4}{\alpha^2}\right)\left(\alpha + \frac{4}{\alpha}\right) = \alpha + \frac{4}{\alpha} + \underbrace{\left(\frac{4}{\alpha} + \frac{16}{\alpha^3}\right)}_{>0}$$

4.2) Per dimostrare che il metodo di Gauss-Seidel è convergente è sufficiente osservare che la matrice $A$ è diagonalmente dominante. Oppure:

$$\det\begin{pmatrix} \alpha\lambda & 0 & \frac{4}{\alpha} \\ 0 & 4\lambda & 0 \\ \frac{4}{\alpha}\lambda & 0 & \alpha\lambda \end{pmatrix} = 4\lambda\left(\alpha^2\lambda^2 - \frac{16}{\alpha^2}\right) = 0 \Longleftrightarrow \lambda_{1,2} = 0, \;\; \lambda_3 = \frac{16}{\alpha^4}$$

$$\rho(B_J) = \frac{16}{\alpha^4} < 1 \Longleftrightarrow |\alpha| > 2$$

4.3)

$$A = M - N, \quad N = M - A = \begin{pmatrix} 0 & 0 & -\frac{4}{\alpha} \\ 0 & \alpha - 4 & 0 \\ -\frac{4}{\alpha} & 0 & 0 \end{pmatrix}$$

$$P = M^{-1}N = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{4}{\alpha} \\ 0 & \alpha - 4 & 0 \\ -\frac{4}{\alpha} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{4}{\alpha^2} \\ 0 & \frac{\alpha-4}{\alpha} & 0 \\ -\frac{4}{\alpha^2} & 0 & 0 \end{pmatrix}$$

$$\|P\|_\infty = \max\Big\{ \underbrace{\frac{\alpha - 4}{\alpha}}_{<1}, \underbrace{\frac{4}{\alpha^2}}_{<1} \Big\} < 1$$

Oppure:

$$\det(P - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & -\frac{4}{\alpha^2} \\ 0 & \frac{\alpha-4}{\alpha} - \lambda & 0 \\ -\frac{4}{\alpha^2} & 0 & -\lambda \end{pmatrix} = 0 \Longleftrightarrow \left(\lambda - \frac{\alpha - 4}{\alpha}\right)\left(\lambda^2 - \frac{16}{\alpha^4}\right) = 0$$

$$\lambda_1 = \frac{\alpha - 4}{\alpha}, \ \lambda_{2,3} = \pm\frac{4}{\alpha^2}, \ \text{con } |\lambda_i| < 1, \ \forall i = 1, 2, 3$$