3) Si consideri la matrice $A \in \mathbb{R}^{n\times n}$ ed il vettore $\mathrm{x} \in \mathbb{R}^n$,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & -1 & \dots & 1 \end{pmatrix}, \quad \mathrm{x} = \begin{pmatrix} 1 \\ 1/2 \\ 1/4 \\ \vdots \\ \vdots \\ 1/2^{n-1} \end{pmatrix}.$$

Calcolare, al variare del parametro n, $A\mathrm{x}$, $\|A\mathrm{x}\|_2$, $\|\mathrm{x}\|_2$. Mostrare che A può essere scritta come prodotto di matrici elementari di Gauss e, tramite questa proprietà, dare una maggiorazione della norma $\|A^{-1}\|_\infty$.

2^ itinere

23/01/2014

M1

$$\|x\|_2 = \sqrt{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \dots \left(\frac{1}{2^{n-1}}\right)^2} = \sqrt{\sum_{i=0}^{n-1} \left(\frac{1}{2^2}\right)^i}$$

$$= \sqrt{\sum_{i=0}^{n-1} \left(\frac{1}{4}\right)^i}$$

Si ricordi che: $\displaystyle\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q}$

$$\Longrightarrow \sum_{i=0}^{n-1} \left(\frac{1}{4}\right)^i = \frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}} = \frac{1-\left(\frac{1}{4}\right)^n}{\frac{3}{4}}$$

$$\|x\|_2 \Longrightarrow \sqrt{\frac{4}{3}\left[1-\left(\frac{1}{4}\right)^n\right]} =$$

$$\frac{2}{\sqrt{3}}\sqrt{1-\left(\frac{1}{4}\right)^n}$$

$$Ax = \begin{bmatrix} 1 \\ -1 + \frac{1}{2} \\ -1 - \frac{1}{2} + \frac{1}{4} \\ -1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} \\ \vdots \\ -1 - \frac{1}{2} - \frac{1}{4} \cdots - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} \end{bmatrix}$$

Elemento i-esimo, $i = 2, \dots, n$

$$(Ax)_i = -\sum_{k=0}^{i-2} \left(\frac{1}{2}\right)^k + \frac{1}{2^{i-1}} = \frac{1}{2^{i-1}} - \frac{1 - \left(\frac{1}{2}\right)^{i-1}}{1 - \frac{1}{2}} =$$

$$\frac{1}{2^{i-1}} - 2\left[1 - \frac{1}{2^{i-1}}\right] = \frac{1}{2^{i-1}} - 2 + \frac{2}{2^{i-1}} =$$

$$\frac{3}{2^{i-1}} - 2 = \frac{3}{\frac{2^i}{2}} - 2 = \frac{6}{2^i} - 2$$

Per $i = 1$, si ottiene $(Ax)_1 = 1$

dunque la formula

$$(Ax)_i = \frac{6}{2^i} - 2$$

vale $\forall\, i = 1, \dots, n$

$$\|Ax\|_2 = \sqrt{\sum_{i=1}^{n}\left[(Ax)_i\right]^2} = \sqrt{\sum_{i=1}^{n}\left(\frac{6}{2^i}-2\right)^2} =$$

$$\sqrt{\sum_{i=1}^{n}\left[\frac{36}{2^{2i}}-\frac{24}{2^i}+4\right]} = \sqrt{36\sum_{i=1}^{n}\left(\frac{1}{4}\right)^i - 24\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i + 4n}$$

$$= \sqrt{36\,\frac{1-\left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}} - 24\,\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} + 4n} =$$

$$= \sqrt{\frac{36.4}{3}\left[1-\left(\frac{1}{4}\right)^{n+1}\right] - 24.2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] + 4n} =$$

$$= \sqrt{48\left[1-\left(\frac{1}{4}\right)^{n+1}\right] - 48\left[1-\left(\frac{1}{2}\right)^{n+1}\right] + 4n} =$$

$$= \sqrt{48\left[-\left(\frac{1}{4}\right)^{n+1} + \left(\frac{1}{2}\right)^{m+1}\right] + 4n}$$