ESERCIZIO 1

1.1) Stima per le spline lineari:

$$|f(x) - S_1(x)| \leq \frac{H^2}{8} \max_{0\leq t\leq 2} |f''(t)|, \qquad H = \frac{2}{M}, \qquad M = \text{numero di intervalli}$$

$$f'(t) = -\frac{1}{\left(t+\frac{1}{2}\right)^2} + 1, \qquad f''(t) = \frac{2}{\left(t+\frac{1}{2}\right)^3}$$

$$\max_{0\leq t\leq 2} |f''(t)| = f''(0) = 16$$

$$\frac{1}{8}\,\frac{4}{M^2}\,16 \leq 10^{-4} \iff M^2 \geq 8\cdot 10^4 \iff M \geq 2\sqrt{2}\;10^2 \approx 282.84 \Longrightarrow M \geq 283.$$

1.2) Metodo dei trapezi compositi.

$$H = \frac{1}{2}, \qquad I_T^C(f) = \frac{1}{2}\cdot\frac{1}{2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + f(2)\right] =$$

$$\frac{1}{4}\left[2 + 2\left(1+\frac{1}{2}\right) + 2\left(\frac{2}{3}+1\right) + 2\left(\frac{1}{2}+\frac{3}{2}\right) + \frac{2}{5} + 2\right] = ... = \frac{221}{60} \approx 3.6833$$

$$I(f) = \int_0^2 f(x)\mathrm{d}x = \ln\left(x+\frac{1}{2}\right) + \frac{x^2}{2}\Big|_0^2 = \ln\frac{5}{2} + 2 - \ln\frac{1}{2} = \ln 5 + 2 \approx 3.6094.$$

Calcolo dell'errore: $|I(f) - I_T^C(f)| \approx 0.0739$.

Stima asintotica dell'errore:

$$\left|\frac{H^2}{12}[f'(0) - f'(2)]\right| = \frac{1}{12}\frac{1}{4}\left|1 - 4 - 1 + \frac{4}{25}\right| = ... = \frac{2}{25} = 0.08.$$

ESERCIZIO 2

2.1) Convergenza del metodo di Jacobi.

$$\det \begin{pmatrix} \alpha\lambda & 4 & 1/2 \\ 0 & 2\alpha\lambda & 1 \\ -1 & 1 & 2\lambda \end{pmatrix} = 0$$

$$\alpha\lambda(4\alpha\lambda^2 - 1) - (4 - \alpha\lambda) = 4\alpha^2\lambda^3 - \cancel{\alpha\lambda} - 4 + \cancel{\alpha\lambda} = 4(\alpha^2\lambda^3 - 1) = 0$$

$$\lambda^3 = \frac{1}{\alpha^2},\ |\lambda| = \sqrt[3]{\frac{1}{\alpha^2}},\ \rho(B_J) = \sqrt[3]{\frac{1}{\alpha^2}} < 1 \Leftrightarrow \frac{1}{\alpha^2} < 1 \Leftrightarrow |\alpha| > 1.$$

2.2) Convergenza del metodo di Gauss-Seidel.

$$\det \begin{pmatrix} \alpha\lambda & 4 & 1/2 \\ 0 & 2\alpha\lambda & 1 \\ -\lambda & \lambda & 2\lambda \end{pmatrix} = 0$$

$$\alpha\lambda(4\alpha\lambda^2 - \lambda) - \lambda(4 - \alpha\lambda) = 4\alpha^2\lambda^3 - \cancel{\alpha\lambda^2} - 4\lambda + \cancel{\alpha\lambda^2} = 4\lambda(\alpha^2\lambda^2 - 1) = 0$$

$$\lambda_1 = 0,\ \lambda_{2,3} = \pm\frac{1}{\alpha},\ \rho(B_{GS}) = \frac{1}{|\alpha|} < 1 \Leftrightarrow |\alpha| > 1.$$

2.3) Matrice di iterazione B_J.

$$-\begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{2\alpha} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 4 & \frac{1}{2} \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \frac{4}{\alpha} & \frac{1}{2\alpha} \\ 0 & 0 & \frac{1}{2\alpha} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$\|B_J\|_1 = \max\left\{\frac{1}{2}, \frac{4}{|\alpha|} + \frac{1}{2}, \cancel{2}\frac{1}{\cancel{2}|\alpha|}\right\} = \frac{4}{|\alpha|} + \frac{1}{2}.$$

2.4) Stima del numero di iterazioni.

$$\alpha = 8, \quad \rho(B_J) = \frac{1}{4}, \quad \rho(B_{GS}) = \frac{1}{8}, \quad k \geq \frac{-\ln\varepsilon}{-\ln\rho(B)}, \quad \varepsilon = 10^{-8}.$$

Jacobi : $\dfrac{-\ln 10^{-8}}{-\ln\frac{1}{4}} \approx 13.2877 \Rightarrow k \geq 14;$ GS : $\dfrac{-\ln 10^{-8}}{-\ln\frac{1}{8}} \approx 8.858475 \Rightarrow k \geq 9.$

ESERCIZIO 3

$$g(x) = \begin{cases} \frac{1}{2}(2^x + x^2 - 1) & x \geq 0 \\ x^2 + 2x & x < 0 \end{cases} \qquad g'(x) = \begin{cases} \frac{1}{2}(2^x \ln 2 + 2x) & x \geq 0 \\ 2x + 2 & x < 0 \end{cases}$$

Punti fissi: $0,\ 1,\ 2.$

$g(0^-) = g(0^+) = 0, \quad g'(0^-) = 2,\ g'(0^+) = (\ln 2)/2 \approx 0.34$

$g(x) = 1 \cap x < 0 : x^2 + 2x = 1 \Rightarrow\ x^2 + 2x - 1 = 0 \Rightarrow\ x = -1 \pm \sqrt{2} \cap x < 0 \Rightarrow x = \alpha = -1 - \sqrt{2}$

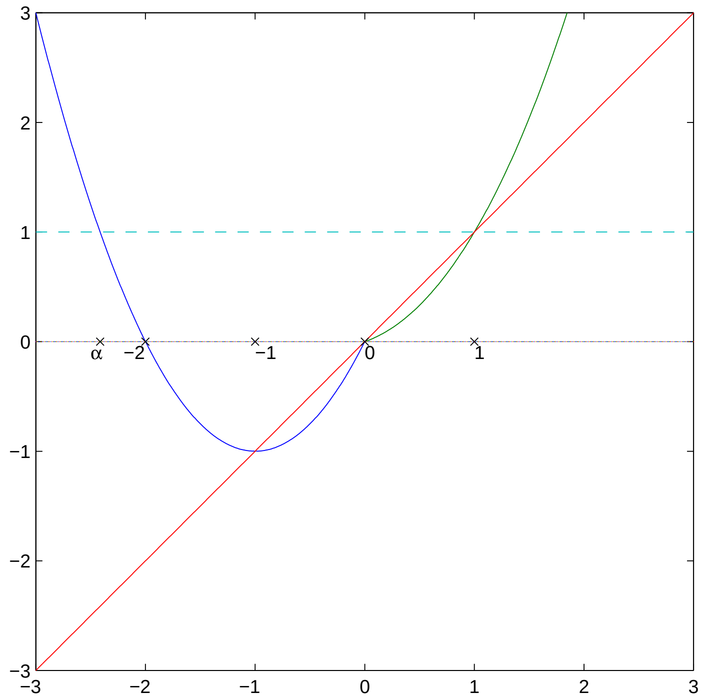

Figure 1: Grafico della funzione $g(x)$.

Studio della convergenza al variare di x_0

$x_0 < \alpha \longrightarrow x_1 > 1;$

$\alpha < x_0 < -2 \longrightarrow 0 < x_1 < 1;$

$-2 < x_0 < -1 \longrightarrow -1 < x_1 < 0;$

$-1 < x_0 < 0 \longrightarrow x_n \searrow -1$
(successione monotona decrescente limitata inferiormente da -1)

$0 < x_0 < 1 \longrightarrow x_n \searrow 0$
(successione monotona decrescente limitata inferiormente da 0)

$x_0 > 1 \longrightarrow x_n \nearrow +\infty$
(successione monotona crescente illimitata superiormente)

Studio dell'ordine.
$g'(-1) = 0$, $g''(x) = 2 \neq 0 \Longrightarrow$ ordine 2;
$g'(0^+) = 0.34 \neq 0 \Longrightarrow$ ordine 1.