CALCOLO NUMERICO 1 (Prova scritta - 19 Giugno 2014)
GIUSTIFICARE LE RISPOSTE

1) Data la funzione $f : \mathbb{R} \to \mathbb{R}$ definita come $f(x) = e^x - x - 2$,

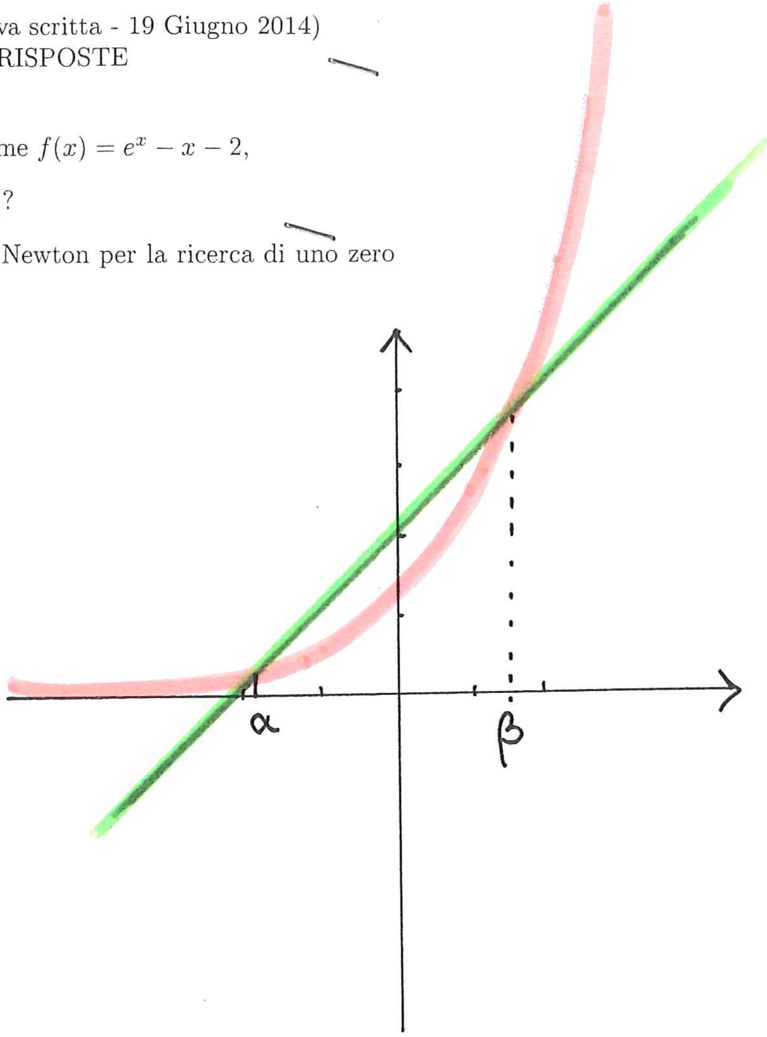
1.1) quanti zeri possiede nell'intervallo $[1, 2]$?

1.2) Discutere l'applicazione del metodo di Newton per la ricerca di uno zero di f nel medesimo intervallo $[1, 2]$.

$f(x) = e^x - x - 2$

$e^x = x + 2$

x	e^x	$x+2$
-2	e^{-2}	> 0
-1	e^{-1}	< 1
0	1	< 2
1	e	< 3
2	e^2	> 4



$-2 < \alpha < -1$ $\qquad$ $1 < \beta < 2$

- $f(1) = e - 3 \approx -0.2817 < 0$
  $f(2) = e^2 - 4 \approx 3.38 > 0$

- $f'(x) = e^x - 1 > 0 \quad x > 0 \quad \Rightarrow \forall x \in [1, 2]$
- $f''(x) = e^x > 0 \quad \forall x \quad \Rightarrow \forall x \in [1, 2]$

$$\left| \frac{f(1)}{f'(1)} \right| = \frac{0.2817}{1.718} < 1 \qquad \left| \frac{f(2)}{f'(2)} \right| = \frac{3.38}{6.38} < 1$$

$\Rightarrow$ m.d.N converge a $\beta \in (1, 2)$ $\forall x_0 \in [1, 2]$

2) Si consideri la formula di quadratura

$$\int_{-1}^{1} f(x)dx \simeq \omega_1 f(-1) + \omega_2 f(-\alpha) + \omega_3 f(\alpha) + \omega_4 f(1),$$

dove ω_1, ω_2, ω_3, ω_4 sono i pesi e $0 < \alpha < 1$.

2.1) Trovare i pesi in funzione di α in modo tale che la formula abbia grado di precisione almeno tre.

2.2) Esistono valori di α per cui la formula ha ordine di precisione $r > 3$? Se sì, calcolare α, i pesi corrispondenti e l'ordine di precisione massimo.

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$$\int_{-1}^{1} f(x)dx \approx \omega_1 f(-1) + \omega_2 f(-\alpha) + \omega_3 f(\alpha) + \omega_4 f(1)$$

) $r=0$ $f(x) = 1$

$$\int_{-1}^{1} 1\, dx = 2 \qquad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 2$$

2) $r=1$ $f(x) = x$

$$\int_{-1}^{1} x\, dx = 0 \qquad -\omega_1 - \alpha\omega_2 + \alpha\omega_3 + \omega_4 = 0$$

3) $r=2$ $f(x) = x^2$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} \qquad \omega_1 1 + \omega_2 \alpha^2 + \omega_3 \alpha^2 + \omega_4 \cdot 1 = \frac{2}{3}$$

4) $r=3$ $f(x) = x^3$

$$\int_{-1}^{1} x^3 dx = 0 \qquad -\omega_1 - \alpha^3\omega_2 + \alpha^3\omega_3 + \omega_4 = 0$$

$$\begin{cases} w_1+w_2+w_3+w_4=2 \\ -w_1-\alpha w_2+\alpha w_3+w_4=0 \\ w_1+w_2\alpha^2+\alpha^2 w_3+w_4=\frac{2}{3} \\ -w_1-\alpha^3 w_2+\alpha^3 w_3+w_4=0 \end{cases}$$

$+2^a - 4^a$

$$/\!/ \quad w_2(-\alpha+\alpha^3)+(\alpha-\alpha^3)w_3=0$$

$$(\alpha^3-\alpha)(w_2-w_3)=0$$

$\alpha \neq 0$

$\alpha \neq \pm 1$ OK per ipotesi $0<\alpha<1$

$$w_2=w_3$$

$$\begin{cases} w_1+2w_2+w_4=2 \\ -w_1+w_4=0 \\ w_1+2w_2\alpha^2+w_4=\frac{2}{3} \\ -w_1+w_4=0 \end{cases} \Longrightarrow \quad w_1=w_4$$

$$\begin{cases} w_1+2w_2+w_1=2 \\ w_1+\alpha^2 w_2=\frac{1}{3} \end{cases}$$

$\Rightarrow \quad w_1+w_2=1 \quad \Rightarrow \quad w_1=1-w_2$

$$1-w_2+\alpha^2 w_2=\frac{1}{3}$$

$$w_2(1-\alpha^2)=\frac{2}{3}$$

$$w_2=\frac{2}{3(1-\alpha^2)}$$

$$\Downarrow$$

$$w_1=1-\frac{2}{3(1-\alpha^2)}=\frac{1-3\alpha^2}{3(1-\alpha^2)}$$

$n = 4$

$$\int_{-1}^{1} f(x)\,dx \approx \frac{1-3\alpha^2}{3(1-\alpha^2)}\Big[f(-1)+f(1)\Big] + \frac{2}{3(1-\alpha^2)}\Big[f(-\alpha)+f(\alpha)\Big]$$

$$\int_{-1}^{1} x^4\,dx = \frac{2}{5}$$

$$\frac{1-3\alpha^2}{3(1-\alpha^2)}(\cancel{1}+1) + \frac{\cancel{2}}{3(1-\alpha^2)}(\alpha^4+\alpha^4) = \frac{\cancel{2}}{5}$$

$$\frac{1-3\alpha^2+2\alpha^4}{3(1-\alpha^2)} = \frac{1}{5}$$

$$5-15\alpha^2+10\alpha^4-3+3\alpha^2$$

$$10\alpha^4-12\alpha^2+2=0$$

$$5\alpha^4-6\alpha^2+1=0$$

$$\alpha^2 = \frac{3\pm 2}{5} \left\langle \begin{array}{l} 1 \quad \text{non acc.} \\ \frac{1}{5} \end{array} \right.$$

$$\alpha = \frac{\sqrt{5}}{5} \qquad w_1 = w_4 = \frac{1-\frac{3}{5}}{3\left(1-\frac{1}{5}\right)} = \frac{2}{5}\cdot\frac{1}{3}\cdot\frac{5}{4} = \frac{1}{6}$$

$$w_2 = w_3 = \frac{2}{3\left(1-\frac{1}{5}\right)} = \frac{2}{3}\cdot\frac{5}{4} = \frac{5}{6}$$

Grado di precisione 5?

$$\int_{-1}^{1} f(x)\,dx \approx \frac{1}{6}f(-1)+\frac{5}{6}f\left(-\frac{\sqrt{5}}{5}\right)+\frac{5}{6}f\left(\frac{\sqrt{5}}{5}\right)+\frac{1}{6}f(1)$$

$$\int_{-1}^{1} x^5\,dx = 0$$

F.Q. $=0$ $\quad$ ($\forall\, x^k$, k dispari $\Rightarrow$ F.Q. esatta)

Grado di precisione 6?

$$\int_{-1}^{1} x^6\,dx = \frac{2}{7}$$

$$\left[\frac{1}{6}\cdot 1+\frac{5}{6}\left(\frac{\sqrt{5}}{5}\right)^6\right]2=\left[\frac{1}{6}+\frac{5}{6}\cdot\frac{5^3}{5^6}\right]2=\left[\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{25}\right]2$$

$$\frac{1}{6}\left(1+\frac{1}{25}\right)2=\frac{1}{\cancel{6}_{3}}\cdot\frac{26}{25}\cdot\cancel{2}=\frac{26}{75}\neq\frac{2}{7}$$

G.P. $r=5$

4) Data la funzione $f(x) = \sin x + 2\cos x$, si determini il polinomio $P \in \mathbb{P}_2$ interpolatore di Lagrange nei nodi $\{0,\ \pi,\ \frac{3\pi}{2}\}$. Dare una maggiorazione dell'errore di interpolazione $E(x) = |f(x) - P(x)|$, con $x \in [0, 2\pi]$. Calcolare l'errore $E(\pi/2)$.

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$$f(x) = \sin x + 2\cos x$$

x_i	0	π	$\frac{3}{2}\pi$
y_i	2	-2	-1

$$L_0(x) = \frac{(x-\pi)(x-\frac{3}{2}\pi)}{(0-\pi)(0-\frac{3}{2}\pi)} = \frac{(x-\pi)(x-\frac{3}{2}\pi)}{\frac{3}{2}\pi^2}$$

$$L_1(x) = \frac{x(x-\frac{3}{2}\pi)}{(\pi)(\pi-\frac{3}{2}\pi)} = \frac{x(x-\frac{3}{2}\pi)}{-\frac{\pi^2}{2}}$$

$$L_2(x) = \frac{x(x-\pi)}{\frac{3}{2}\pi \cdot (\frac{3}{2}\pi - \pi)} = \frac{x(x-\pi)}{\frac{3}{4}\pi^2}$$

$$P_2(x) = \frac{4}{3\pi^2}(x-\pi)(x-\frac{3}{2}\pi) +$$

$$2 \cdot \frac{2}{\pi^2}\, x(x-\frac{3}{2}\pi) +$$

$$- \frac{4}{3\pi^2}(x)(x-\pi) = \frac{4}{\pi^2}x^2 - \frac{8}{\pi}x + 2$$

$$|f(x) - P_2(x)| \leq \frac{1}{3!} \underbrace{\max_{0 \leq t \leq 2\pi} |\omega(t)|}_{A} \underbrace{\max_{0 \leq t \leq 2\pi} |f^{(3)}(t)|}_{B}$$

A: $\max_{0 \leq t \leq 2\pi} |t| \, |t - \pi| \, |t - \frac{3}{2}\pi| \leq$

$$2\pi \cdot \pi \cdot \frac{3}{2}\pi = 3\pi^3$$

B $f'(t) = \cos t - 2 \operatorname{sen} t$

$f''(t) = -\operatorname{sen} t - 2\cos t$

$f'''(t) = -\cos t + 2 \operatorname{sen} t$

$$|f'''(t)| \leq 1 + 2 = 3$$

$$|f(x) - p_2(x)| \leq \frac{1}{6} \cdot 3\pi^3 \cdot 3 = \frac{3}{2}\pi^3$$

$f\left(\frac{\pi}{2}\right) = 1$

$$p_2\left(\frac{\pi}{2}\right) = \frac{\cancel{4}^{2}}{3\cancel{\pi^2}}\left(-\frac{\cancel{\pi}}{\cancel{2}}\right)(-\pi) + \frac{4}{\cancel{\pi^2}} \cdot \frac{\cancel{\pi}}{2}(-\pi) - \frac{4}{3\cancel{\pi^2}}\left(\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) =$$

$$\frac{2}{3} - 2 + \frac{1}{3} = -1$$

$$E\left(\frac{\pi}{2}\right) = 2$$

3) Sia $\alpha \in \mathbb{R}$, e sia A la matrice

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$$A = \begin{pmatrix} \alpha+4 & \alpha \\ \alpha & \alpha+4 \end{pmatrix}.$$

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Per i valori di α per cui A è invertibile calcolare i valori dei numeri di condizionamento $K_1(A)$ e $K_2(A)$.

$$\det A = \alpha^2 + 8\alpha + 16 - \alpha^2 = 8\alpha + 16 \neq 0$$

$$A = \begin{pmatrix} \alpha+4 & \alpha \\ \alpha & \alpha+4 \end{pmatrix}$$

$$\Downarrow$$

$$A^{-1} = \frac{1}{8\alpha+16} \begin{bmatrix} \alpha+4 & -\alpha \\ -\alpha & \alpha+4 \end{bmatrix} \qquad \alpha \neq -2$$

$$\|A\|_1 = |\alpha+4| + |\alpha| = \begin{cases} -\alpha-4-\alpha = -2\alpha-4 & \alpha \leq -4 \\ \alpha+4-\alpha = 4 & -4 < \alpha < 0 \\ 2\alpha+4 & \alpha \geq 0 \end{cases}$$

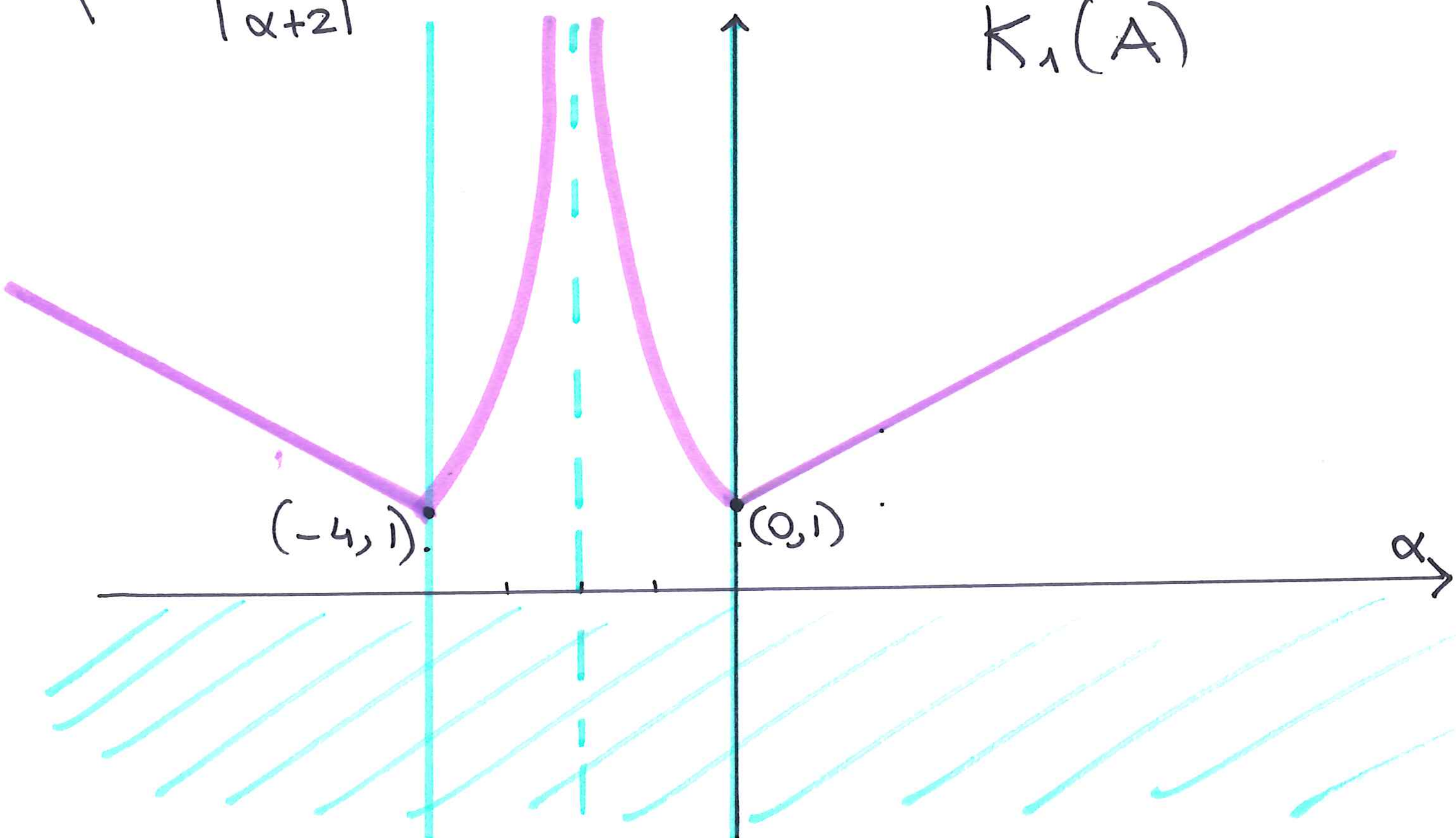
$$\|A^{-1}\|_1 = \frac{1}{|8\alpha+16|} \left(|\alpha+4| + |\alpha| \right)$$

$$K_1(A) = \frac{(|\alpha+4| + |\alpha|)^2}{8|\alpha+2|} = \frac{\alpha^2 + 8\alpha + 16 + \alpha^2 + 2|\alpha(\alpha+4)|}{8|\alpha+2|}$$

$$\frac{\alpha^2+4\alpha+8+|\alpha(\alpha+4)|}{4|\alpha+2|} =$$

$$\begin{cases} \dfrac{\alpha^2+4\alpha+8+\alpha^2+4\alpha}{4|\alpha+2|} & \alpha \leq -4 \cup \alpha \geq 0 \\ \dfrac{\alpha^2+4\alpha+8-\alpha^2-4\alpha}{4|\alpha+2|} & -4 < \alpha < 0 \quad \wedge \quad \alpha \neq -2 \end{cases}$$

$$\begin{cases} \dfrac{2\alpha^2+8\alpha+8}{4|\alpha+2|} = \dfrac{2(\alpha+2)^2}{4|\alpha+2|} = \dfrac{|\alpha+2|}{2} & \alpha \leq -4 \cup \alpha \geq 0 \\ \dfrac{2}{|\alpha+2|} & -4 < \alpha < 0 \quad \wedge \quad \alpha \neq -2 \end{cases}$$



$$K_2(A) = \frac{\max|\lambda(A)|}{\min|\lambda(A)|}$$

[A simmetrica

$\Rightarrow \|A\|_2 = \max|\lambda(A)|$

$\|A^{-1}\|_2 = \dfrac{1}{\min|\lambda(A)|}$]

$$\det\begin{bmatrix} \alpha+4-\lambda & \alpha \\ \alpha & \alpha+4-\lambda \end{bmatrix} = 0$$

$(\alpha+4-\lambda)^2 - \alpha^2 = 0$

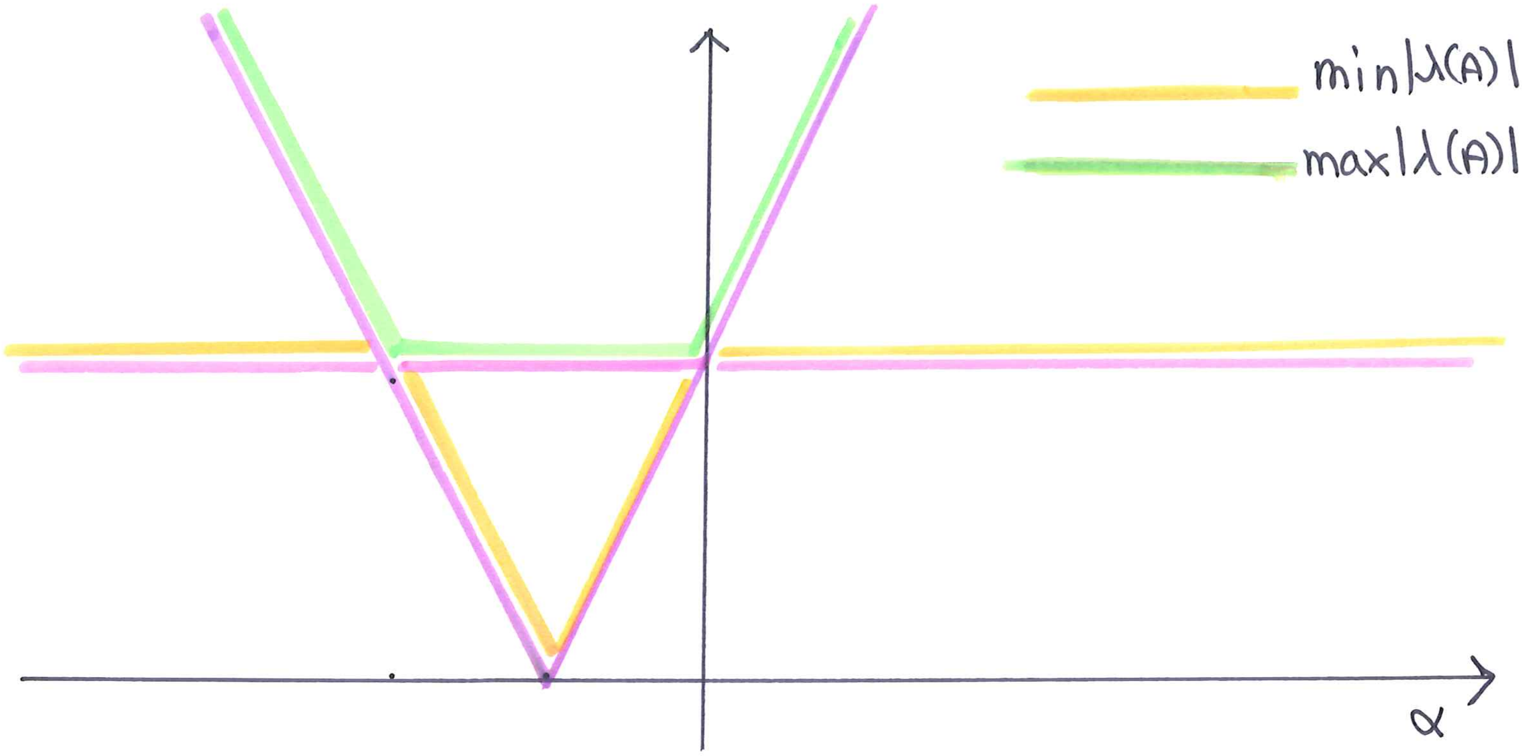
$(\alpha+4-\lambda)^2 = \alpha^2 \quad \begin{cases} \cancel{\alpha}+4-\lambda = \cancel{\alpha} & \lambda = 4 \\ \alpha+4-\lambda = -\alpha & \lambda = 2\alpha+4 \end{cases}$

$\max|\lambda(A)| = \begin{cases} 4 & \text{se } |2\alpha+4| \leq 4 \quad -4 \leq \alpha \leq 0 \\ |2\alpha+4| & \text{se } |2\alpha+4| > 4 \quad \alpha \leq -4 \vee \alpha \geq 0 \end{cases}$

- $2\alpha+4 \leq 4 \qquad -4 \leq 2\alpha+4 \leq 4$

  $-8 \leq 2\alpha \leq 0 \qquad -4 \leq \alpha \leq 0$

$\min|\lambda(A)| = \begin{cases} 4 & \text{se } |2\alpha+4| \geq 4 \quad -4 \leq \alpha \leq 0 \\ |2\alpha+4| & \text{se } |2\alpha+4| \leq 4 \quad \alpha \leq -4 \vee \alpha \geq 0 \end{cases}$



$$K_2(A) = \frac{|2\alpha+4|}{4} = \frac{|\alpha+2|}{2}, \quad \alpha \leq -4 \quad \vee \quad \alpha \geq 0$$

$$K_2(A) = \frac{4}{|2\alpha+4|} = \frac{2}{|\alpha+2|} \quad -4 \leq \alpha \leq 0 \quad \alpha \neq -2$$

$$K_1(A) = K_2(A) \quad \forall \ \alpha \neq -2$$