Calcolo Numerico 1 – 18/09/2012 – 

1.1) $A^{-1}$ (si calcola facilmente ricavando le equazioni da $AA^{-1} = I$, elemento per elemento)

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 & \dots & -1-\alpha \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & & 0 & 1 & & \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & \dots & & 0 & 1 \end{bmatrix} \qquad \boxed{\alpha > 0}$$

$\|A\|_\infty = (N-1)\cdot 1 + 1 + \alpha = N + \alpha$

$\|A^{-1}\|_\infty = N + \alpha$

$K(A_N) = (N+\alpha)^2$

1.2) Jacobi

$$\det \begin{bmatrix} \lambda & +1 & \dots & 1+\alpha \\ 0 & \lambda & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots 0 & \lambda \end{bmatrix} = \lambda^N = 0 \qquad \lambda = 0$$

$\rho(B_J) = 0$

Gauss-Seidel, in modo analogo $\rho(B_{GS}) = 0$

$R(B_J) = R(B_{GS}) = -\ln \rho(B_J) = -\ln \rho(B_{GS}) \to \infty$

Convergono in 2 iterazioni.

2) $f(x) = \log(3+x) \qquad -1 \leq x \leq 1$

$P_N \in \mathbb{P}_N \qquad x_K =$ nodi di Chebyshev

Stima

$$|f(x) - P_N(x)| \leq \frac{|(x-x_0)(x-x_1)\dots(x-x_N)|}{(N+1)!} |f^{N+1}(t)| \qquad t \in [-1,1]$$

Maggiorazione per il 2° membro:

$$\leq \frac{\overbrace{2 \cdot 2 \cdot \dots \cdot 2}^{N+1 \text{ volte}}}{(N+1)!} \cdot \max_{-1 \leq t \leq 1} |f^{N+1}(t)|$$

$$f(t) = \log(3+t)$$

$$f'(t) = \frac{1}{3+t}$$

$$f''(t) = -\frac{1}{(3+t)^2}$$

$$f'''(t) = \frac{2}{(3+t)^3}$$

$$f^{IV}(t) = -\frac{6}{(3+t)^4}$$

$$\vdots$$

$$f^{N+1}(t) = \pm \frac{N!}{(3+t)^{N+1}} \Rightarrow \max_{-1 \leq t \leq 1} |f^{N+1}(t)| \leq \frac{N!}{2^{N+1}}$$

Secondo membro $\leq$

$$\frac{2^{N+1}}{(N+1)\,N!} \cdot \frac{N!}{2^{N+1}} = \frac{1}{N+1} \xrightarrow[N \to \infty]{} 0$$

3) $I=\int_0^1 f(x)dx \approx A_1 f(a)+A_2 f(1-a) \quad 0\le a<1$

$r=0 \quad f=1 \quad \int_0^1 1\,dx = 1$

$\boxed{A_1+A_2=1} \longrightarrow A_1 = 1-A_2$

$r=1 \quad f=x \quad \int_0^1 x\,dx = \frac{1}{2}$

$\boxed{A_1\cdot a + A_2(1-a) = \frac{1}{2}} \longrightarrow (1-A_2)a + A_2 - A_2 a = \frac{1}{2}$

$a - 2aA_2 + A_2 = \frac{1}{2}$

$r=2 \quad f=x^2 \quad \int_0^1 x^2\,dx = \frac{1}{3}$

$\boxed{A_1\cdot a^2 + A_2(1-a)^2 = \frac{1}{3}} \quad (1-A_2)a^2 + A_2(1-2a+a^2) = \frac{1}{3}$

$$\begin{cases} A_1 = 1-A_2 \\ a + A_2 - 2aA_2 = \frac{1}{2} \\ a^2 - \cancel{A_2 a^2} + A_2 - 2aA_2 + \cancel{A_2 a^2} = \frac{1}{3} \end{cases}$$

$$\begin{cases} A_1 = 1-A_2 \\ a + A_2 - 2aA_2 = \frac{1}{2} \\ a^2 + A_2 - 2aA_2 = \frac{1}{3} \end{cases}$$

$a - a^2 \quad / \quad / \quad = \frac{1}{6}$

$$\begin{cases} A_1 = 1-A_2 \\ a + A_2 - 2a A_2 = \frac{1}{2} \\ 6a^2 - 6a + 1 = 0 \end{cases}$$

$a = \frac{3 \pm \sqrt{9-6}}{6} = \frac{3\pm\sqrt{3}}{6} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$

Soluzioni simmetriche

$$a = \frac{1}{2} - \frac{\sqrt{3}}{6}$$

$$\begin{cases} \cancel{\frac{1}{2}} - \frac{\sqrt{3}}{6} + A_2 - 2\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) A_2 = \cancel{\frac{1}{2}} \\ -\frac{\sqrt{3}}{6} + \cancel{A_2} - \cancel{A_2} + \frac{\sqrt{3}}{3} A_2 = 0 \end{cases}$$

Nella $2^a$

$$-\sqrt{3} + 2A_2\sqrt{3} = 0 \qquad A_2 = \frac{1}{2}$$

$$A_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$a = \frac{1}{2} + \frac{\sqrt{3}}{6}$$

$$\cancel{\frac{1}{2}} + \frac{\sqrt{3}}{6} + A_2 - 2\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) A_2 = \cancel{\frac{1}{2}}$$

$$\frac{\sqrt{3}}{6} + \cancel{A_2} - \cancel{A_2} - \frac{\sqrt{3}}{3} A_2 = 0 \qquad A_2 = \frac{1}{2}$$

$$a = \frac{1}{2} - \frac{\sqrt{3}}{6} \qquad 1 - a = \frac{1}{2} + \frac{\sqrt{3}}{6}$$

$$a = \frac{1}{2} + \frac{\sqrt{3}}{6} \qquad 1 - a = \frac{1}{2} - \frac{\sqrt{3}}{6}$$

FQ: $\frac{1}{2} f\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)$

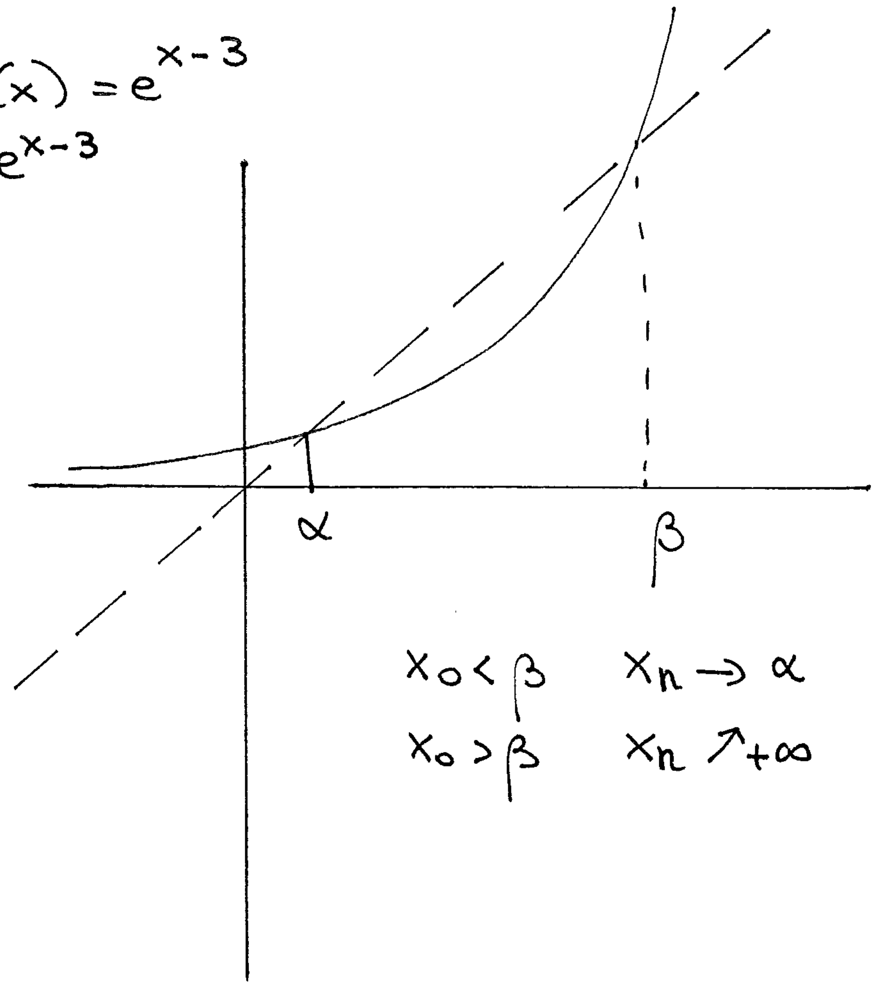
4) $x_{n+1} = e^{x_n - 3}$ $\qquad g(x) = e^{x-3}$

$f(x) = x - e^{x-3}$

| $x$ | | $g(x)$ |
|---|---|---|
| 0 | < | $\frac{1}{e^3}$ |
| 1 | > | $\frac{1}{e^2}$ |
| 2 | > | $\frac{1}{e}$ |
| 3 | > | 1 |
| 4 | > | $e$ |
| 5 | < | $e^2$ |

$\alpha \in (0,1)$

$\beta \in (4;5)$




$x_0 < \beta \quad x_n \to \alpha$

$x_0 > \beta \quad x_n \nearrow +\infty$

Metodo di Newton $\quad x_{n+1} = x_n - \dfrac{x_n - e^{x_n-3}}{1 - e^{x_n-3}} = \dfrac{-x_n e^{x_n-3} + e^{x_n-3}}{1 - e^{x_n-3}}$

$f \in C^2[4;5]$, $\forall x_0 \in [4,5]$ $x_n \to \beta$. Infatti valgono:

1) $f(4) = 4 - e > 0 \qquad f(5) = 5 - e^2 < 0$

2) $f'(x) = 1 - e^{x-3} > 0 \quad e^{x-3} < 1 \quad x < 3 \Rightarrow f'(x) < 0 \quad x \in [4;5]$

3) $f''(x) = -e^{x-3} < 0 \quad x \in [4;5]$

$\left|\dfrac{f(4)}{f'(4)}\right| = \dfrac{4-e}{e-1} < 1 \qquad 4 - e < e - 1 \qquad 5 < 2e \qquad$ sì

$\left|\dfrac{f(5)}{f'(5)}\right| = \left|\dfrac{5-e^2}{1-e^2}\right| = \dfrac{e^2-5}{e^2-1} < 1 \qquad \cancel{e^2} - 5 < \cancel{e^2} - 1 \qquad$ sì

In modo analogo. Si consideri l'intervallo $[-1, 1]$.

$$f(-1) = -1 - e^{-4} = -\left(1 + \frac{1}{e^4}\right) = -\frac{e^4+1}{e^4}$$

$$f(1) = 1 - e^{-2} = 1 - \frac{1}{e^2} = \frac{e^2-1}{e^2}$$

$$f'(x) = 1 - e^{x-3} > 0 \quad x < 3 \Rightarrow f'(x) > 0 \quad x \in [-1, 1]$$

$$f''(x) = -e^{x-3} < 0 \quad \forall x$$

$$f'(-1) = 1 - e^{-4} = 1 - \frac{1}{e^4} = \frac{e^4-1}{e^4}$$

$$f'(1) = 1 - e^{-2} = 1 - \frac{1}{e^2} = \frac{e^2-1}{e^2}$$

$$\left|\frac{f(-1)}{f'(-1)}\right| = \frac{e^4+1}{e^4} \cdot \frac{e^4}{e^4-1} = \frac{e^4+1}{e^4-1} < 2 \qquad e^4+1 < 2e^4-2$$

$$3 < e^4 \quad \text{sì}$$

$$\left|\frac{f(1)}{f'(1)}\right| = \frac{e^2-1}{e^2} \cdot \frac{e^2}{e^2-1} = 1 < 2 \quad \text{sì}$$

$\forall x_0 \in [-1, 1] \quad x_n \to \alpha$.