1) Data la funzione $f(x) = x^2 - 3x + 2$ studiare al variare di $x_0 \in \mathbb{R}$ la convergenza e l' ordine dei metodi iterativi

$$x_{n+1} = \frac{x_n^2 + 2}{3}, \quad x_{n+1} = x_n^2 - 2x_n + 2,$$

per l'approssimazione degli zeri di f.

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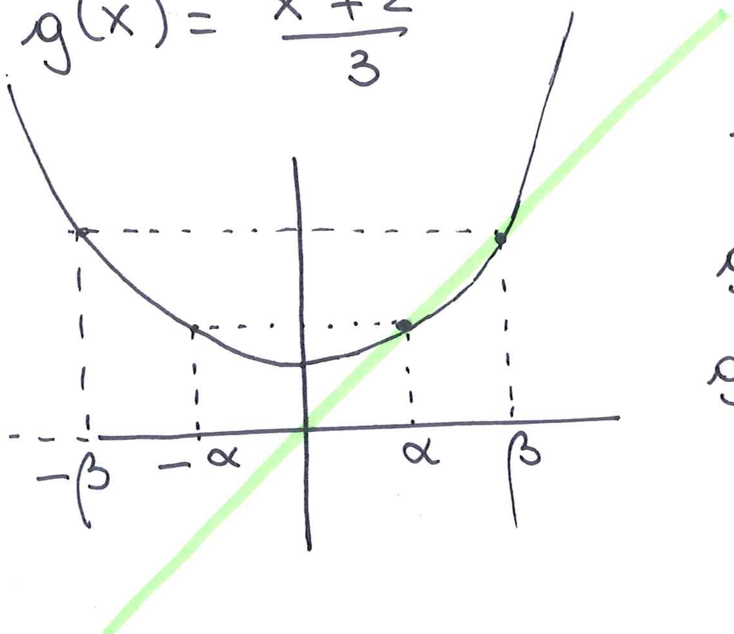
$$x_{n+1} = \frac{x_n^2 + 2}{3}$$

$$x = \frac{x^2+2}{3} \Rightarrow x^2 - 3x + 2 = 0$$

$\alpha = 1 \quad \beta = 2$

$$g(x) = \frac{x^2+2}{3}$$



$g'(x) = \frac{2}{3}x$

$g'(\alpha) = \frac{2}{3} < 1$

$g'(\beta) = \frac{4}{3} > 1$

$x_0 < -\beta \quad x_1 > \beta$

$x_0 = -\beta \quad$ "$x_K = \beta$" $\forall K \geq 1$

$-\beta < x_0 < -\alpha \quad \alpha < x_1 < \beta$

$x_0 = -\alpha \quad$ "$x_K = \alpha$" $\forall K \geq 1$

$-\alpha < x_0 < 0 \quad 0 < x_1 < \alpha$

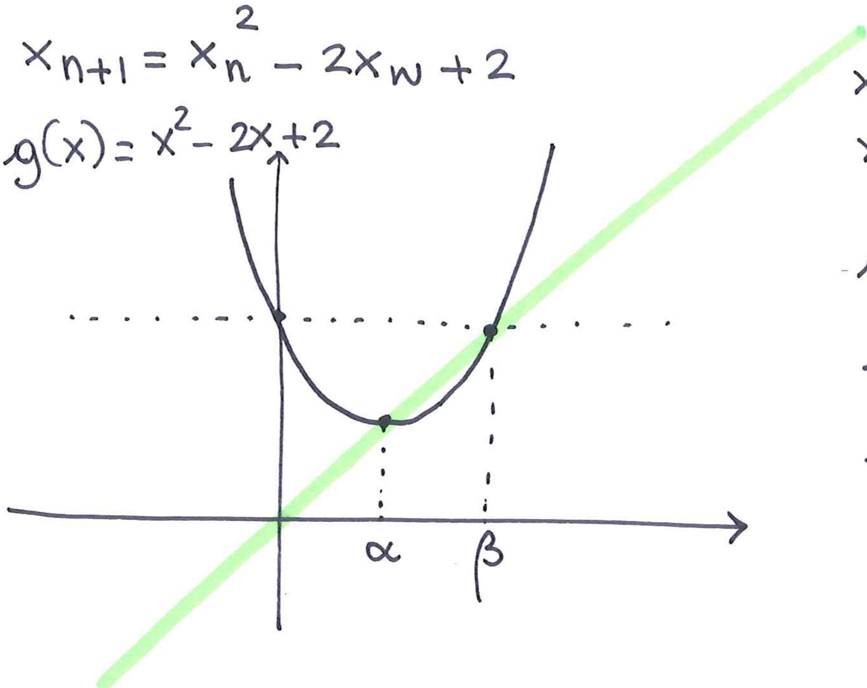
- $0 \leq x_0 < \alpha$ SUCC. MON. CRESC. LIM. SUP DA α: $x_K \nearrow \alpha$
- $x_0 = \alpha$ "$x_K = \alpha$" $\forall K \geq 1$
- $\alpha < x_0 < \beta$ SUCC. MON. DECR. LIM. INF DA α: $x_K \searrow \alpha$
- $x_0 = \beta$ "$x_K = \beta$" $\forall K \geq 1$
- $x_0 > \beta$ SUCC. MON. CRESC. ILLIM. SUP. $x_K \nearrow +\infty$

RIEPILOGO CONVERGENZA e ORDINE metodo 1

$-\beta < x_0 < \beta \quad x_K \Rightarrow \alpha$: ordine 1

$x_{n+1} = x_n^2 - 2x_n + 2$

$g(x) = x^2 - 2x + 2$



$x = x^2 - 2x + 2$

$x^2 - 3x + 2 = 0$

$g'(x) = 2x - 2$

$g'(\alpha) = 0 \qquad g''(x) = 2 \neq 0$

$g'(\beta) = 2$

- $x_0 < 0 \qquad x_1 > \beta$
- $0 < x_0 < \alpha \qquad \alpha < x_1 < \beta$
- $x_0 = \alpha \qquad x_1 = \alpha \quad \forall K \geq 1$
- $\alpha < x_0 < \beta$ SUCC. MON. DECR. LIM. INF. DA α: $x_K \searrow \alpha$
- $x_0 = \beta \quad$ " $x_K = \beta$ " $\forall K \geq 1$
- $x_0 > \beta$ SUCC. MON. CRESC. ILLIM. SUP. $x_K \nearrow +\infty$

RIEPILOGO CONVERGENZA e ORDINE metodo 2

$0 < x_0 < \beta \; : \; x_K \Rightarrow \alpha$: ordine 2

1

2) Assegnati i valori reali $a < b$, trovare ω_1 e ω_2 in modo tale che la formula di quadratura,

$$\int_a^b f(x)dx \simeq \omega_1 f\left(a + \frac{b-a}{3}\right) + \omega_2 f\left(a + 2\frac{b-a}{3}\right)$$

abbia grado di precisione massimo. Quale è il grado di precisione della formula ottenuta?

18/9/2014

$$\int_a^b f(x)\,dx \approx \omega_1 f\left(a + \frac{b-a}{3}\right) + \omega_2 f\left(a + 2\,\frac{b-a}{3}\right)$$

$r=0 \qquad \omega_1 + \omega_2 = b - a$

$r=1 \qquad \int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$

$$\omega_1\left(a + \frac{b-a}{3}\right) + \omega_2\left(a + 2\,\frac{b-a}{3}\right)$$

$$\underbrace{\omega_1 a + \omega_1 \frac{b-a}{3} + \omega_2 a}\, + 2\omega_2 \frac{b-a}{3} = \frac{1}{2}(b+a)(b-a)$$

$$(\omega_1+\omega_2)\,a + \left(\frac{1}{3}\omega_1 + \frac{2}{3}\omega_2\right)\cancel{(b-a)} = \frac{1}{2}(b+a)\cancel{(b-a)}$$

$\cancel{(b-a)}$

$$a + \frac{1}{3}\omega_1 + \frac{2}{3}\omega_2 = \frac{1}{2}(b+a)$$

$$\frac{1}{3}\omega_1 + \frac{2}{3}\omega_2 = \frac{b+a-2a}{2}$$

$$\frac{1}{3}\omega_1 + \frac{2}{3}\omega_2 = \frac{b-a}{2} \qquad \omega_1 = b-a-\omega_2$$

$$\frac{1}{3}(b-a) - \frac{1}{3}\omega_2 + \frac{2}{3}\omega_2 = \frac{b-a}{2}$$

$$\frac{1}{3}\omega_2 = (b-a)\left(\frac{1}{2}-\frac{1}{3}\right) \Rightarrow \omega_2 = \frac{b-a}{6}\cdot 3 = \frac{b-a}{2}$$

$$\Downarrow$$

$$\omega_1 = \frac{b-a}{2}$$

$$\int_a^b f(x)\,dx \cong \frac{b-a}{2}\left[f\left(a+\frac{b-a}{3}\right)+f\left(a+\frac{2}{3}(b-a)\right)\right]$$

La formula, per costruzione, ha grado ≥ 1.

Controllo: $r=2$ $\quad f(x)=x^2$

$$\int_a^b x^2\,dx = \left.\frac{x^3}{3}\right|_a^b = \frac{1}{3}(b^3-a^3) = \frac{1}{3}(b-a)(a^2+ab+b^2)$$

FQ

$$\frac{b-a}{2}\left[\left(a+\frac{b-a}{3}\right)^2+\left(a+\frac{2}{3}(b-a)\right)^2\right]=$$

$$\frac{b-a}{2}\left[\left(\frac{2a+b}{3}\right)^2+\left(\frac{a+2b}{3}\right)^2\right]=$$

$$\frac{b-a}{2}\left[\frac{4a^2+4ab+b^2+a^2+4ab+4b^2}{9}\right]=$$

$$\frac{b-a}{2}\left[\frac{5a^2+8ab+5b^2}{9}\right]=\frac{1}{3}(b-a)\,\frac{5a^2+8ab+5b^2}{6}$$

$\neq$

$\Downarrow$

Grado di precisione $r=1$

OSS

FQ 2 punti aperta $\approx$ FQ 2 punti chiusa
N.C. $\qquad$ N.C. (Trapezi, $r=1$)

3) Stimare il numero minimo di sottointervalli di uguale ampiezza in cui si deve suddividere l'intervallo $[-1, 1]$, affinchè l'errore che si commette interpolando con una spline lineare la funzione $f(x) = x^2 + e^x + e^{-x}$ sia minore di 10^{-3}.

M1

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$[-1,1] \qquad f(x) = x^2 + e^x + e^{-x}$

$$\|f - S_1\|_\infty \le \frac{h^2}{8} \underbrace{\max_{-1\le x\le 1} |f''(x)|}_{M} \qquad h = \frac{2}{n}$$

$$\frac{1}{8}\left(\frac{2}{n}\right)^2 M < 10^{-3}$$

$$f'(x) = 2x + e^x - e^{-x}$$

$$f''(x) = 2 + e^x + e^{-x}$$

$$f'''(x) = e^x - e^{-x} = e^x - \frac{1}{e^x} \ge 0 \qquad x \ge 0$$

0

$$\left(f''(0) = 4\right) \qquad f''(1) = f''(-1) = 2 + e + \frac{1}{e}$$

$$\frac{1}{8}\frac{4}{n^2} \cdot 5{,}086 = \frac{}{n^2} < 10^{-3} \qquad n^2 > 2543$$

$$n > 50{,}42\ldots \qquad \bar{n} = 51$$

$$f'' = 2 + e^x + e^{-x} < 2 + 2e \qquad x \in [-1,1]$$

$$\frac{1}{8} \cdot \frac{4}{n^2} \cdot 2(1+e) = \frac{1+e}{n^2} < \frac{1}{1000} \qquad n > \sqrt{1000(1+e)} \approx 60.97$$

$$\bar{n} = 61$$

4) Dato il sistema lineare $A\mathbf{x} = \mathbf{b}$, con

$$A = \begin{pmatrix} \alpha & -1 & 0 \\ -1 & 1-\alpha & -1 \\ 0 & -1 & 1+\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}:$$

4.1) determinare per quali valori del parametro α la matrice è diagonalmente dominante;

4.2) calcolare e rappresentare graficamente la quantità $\|A\|_\infty$ al variare di $\alpha \in \mathbb{R}$;

4.3) stabilire se per $\alpha = \frac{1}{2}$ i metodi di Jacobi e Gauss-Seidel convergono, motivando la risposta.

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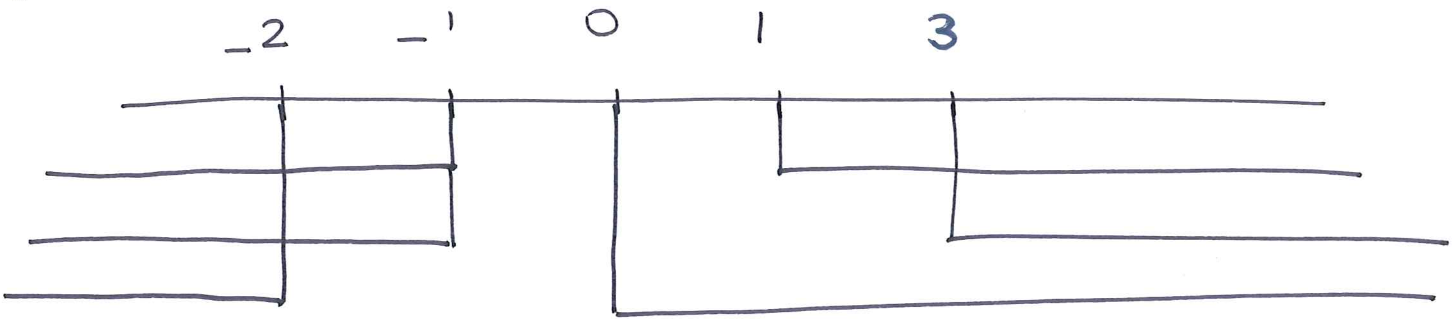
$$A = \begin{pmatrix} \alpha & -1 & 0 \\ -1 & 1-\alpha & -1 \\ 0 & -1 & 1+\alpha \end{pmatrix} \qquad \alpha \in \mathbb{R}$$

D.D.

$$\begin{cases} |\alpha| > 1 \\ |1-\alpha| > 2 \\ |1+\alpha| > 1 \end{cases} \qquad \begin{cases} \alpha < -1 \ \cup \ \alpha > 1 \\ \alpha - 1 < -2 \ \cup \ \alpha - 1 > 2 \\ \alpha + 1 < -1 \ \cup \ \alpha + 1 > 1 \end{cases}$$

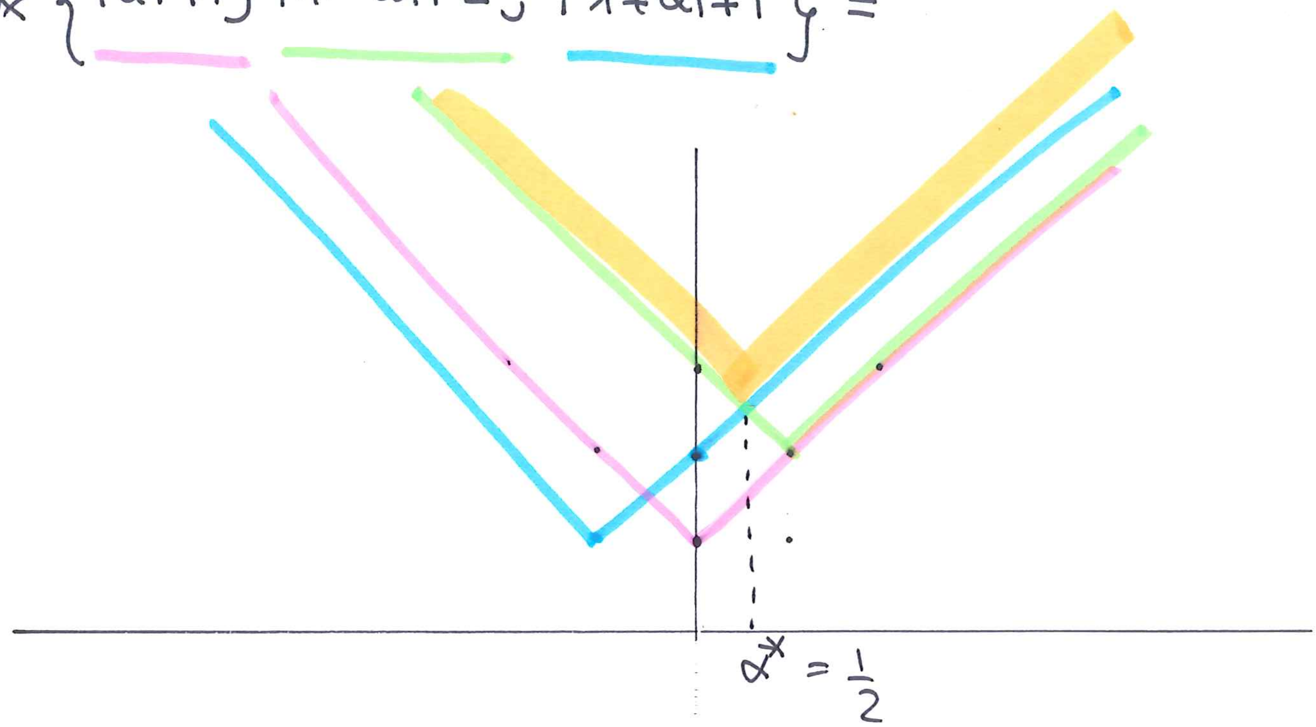
$$\begin{cases} \alpha < -1 \ \cup \ \alpha > 1 \\ \alpha < -1 \ \cup \ \alpha > 3 \\ \alpha < -2 \ \cup \ \alpha > 0 \end{cases}$$



D.D. $\quad \alpha < -2 \ \cup \ \alpha > 3$

$\boxed{\|A\|_\infty} =$

$\max\{\underline{|\alpha|+1}, \underline{|1-\alpha|+2}; \underline{|1+\alpha|+1}\} =$



$1-\alpha+2 = 1+\alpha+1$

$\alpha^* = \frac{1}{2}$

$$\|A\|_\infty = \begin{cases} |1-\alpha|+2 = -\alpha+3 & \alpha \le \frac{1}{2} \\ |1+\alpha|+1 = \alpha+2 & \alpha > \frac{1}{2} \end{cases}$$

$\alpha = \frac{1}{2}$

$$\begin{pmatrix} \frac{1}{2} & -1 & 0 \\ -1 & \frac{1}{2} & -1 \\ 0 & -1 & \frac{3}{2} \end{pmatrix}$$

Jacobi

$$\begin{pmatrix} \frac{1}{2}\lambda & -1 & 0 \\ -1 & \frac{1}{2}\lambda & -1 \\ 0 & -1 & \frac{3}{2}\lambda \end{pmatrix} =$$

$$\frac{1}{2}\lambda\left(\frac{3}{4}\lambda^2 - 1\right) + \left(-\frac{3}{2}\lambda\right) = \frac{3}{8}\lambda^3 - 2\lambda = 0$$

$\lambda = 0 \qquad \frac{3}{8}\lambda^2 - 2 = 0 \qquad \lambda^2 = \frac{16}{3} \qquad \lambda = \pm\frac{4}{\sqrt{3}}$

$\rho(B_J) = \frac{4}{\sqrt{3}} > 1$ Non converge

Matrice tridiagonale:

$\rho(B_{GS}) = \frac{16}{3} > 1$

(convergono o divergono simultaneamente

$$\rho^2(B_J) = \rho(B_{GS})$$