CALCOLO NUMERICO 1 (22 Gennaio 2015) - Seconda prova in itinere

- GIUSTIFICARE LE RISPOSTE -

1) Data la funzione $f(x) = x^3 - x$, studiare al variare di $x_0 \in \mathbb{R}$ la convergenza e l' ordine del metodo iterativo

$$x_{n+1} = \frac{x_n^3 + 3x_n}{3x_n^2 + 1}.$$

$$x = \frac{x^3+3x}{3x^2+1} \qquad 3x^3 + x = x^3 + 3x \qquad 2x^3 - 2x = 0 \qquad x(x^2-1) = 0 \qquad \begin{array}{l} x=0 \\ x=\pm 1 \end{array}$$

Studio di $y = g(x)$. C.E. $\mathbb{R}$ Funzione dispari $g(0)=0$

$g(\pm 1) = \pm 1$

Asintoto obliquo

$$m = \frac{1}{3} \qquad \lim_{x\to\infty} \frac{x^3+3x}{3x^2+1} - \frac{1}{3}x = \lim_{x\to\infty} \frac{3x^3 + 9x - 3x^3 - x}{3x^2+1} = 0 \Rightarrow y = \frac{1}{3}x$$

$$g'(x) = \frac{(3x^2+3)(3x^2+1) - 6x(x^3+3x)}{(3x^2+1)^2} = \frac{9x^4 + 9x^2 + 3x^2 + 3 - 6x^4 - 18x^2}{(3x^2+1)^2} = \frac{3x^4 - 6x^2 + 3}{(3x^2+1)^2}$$

$$= \frac{3(x^2-1)^2}{(3x^2+1)^2} \quad \begin{array}{ll} >0 & \forall x \neq \pm 1 \\ =0 & x = \pm 1 \ \text{(moltepl. 2)} \end{array}$$

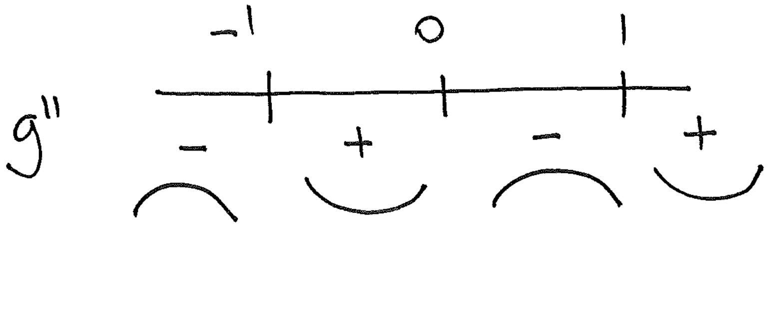
$(1,1)$, $(-1,-1)$ flessi a tangente orizzontale

c.s. $g'(\pm 1) = 0$; $g'(0) = 3 > 1$

$$g''(x) = 3 \cdot \frac{2(x^2-1) \cdot 2x (3x^2+1)^2 - (x^2-1)^2 \cdot 2(3x^2+1) \cdot 6x}{(3x^2+1)^4} =$$

$$= 3 \cdot \frac{4x(3x^2+1)(x^2-1)\left[(3x^2+1) - 3(x^2-1)\right]}{(3x^2+1)^4} =$$

$$= \frac{12\, x(x^2-1)(\cancel{3x^2}+1-\cancel{3x^2}+3)}{(3x^2+1)^3} = 48 \frac{x(x^2-1)}{(3x^2+1)^3} > 0$$

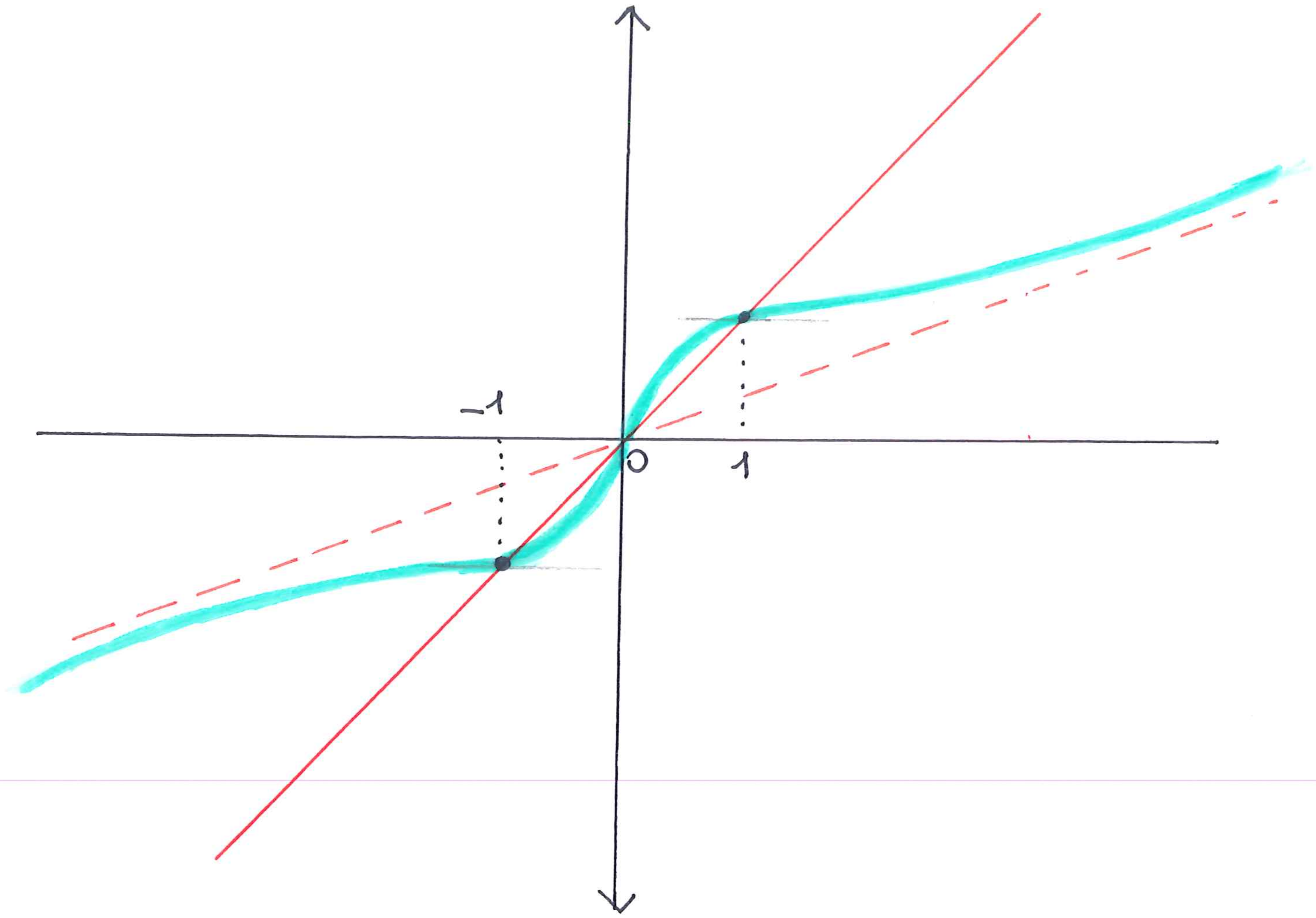


$g''(\pm 1) = 0$

moltepl. 1

$\Downarrow$

$g'''(\pm 1) \neq 0$



$x_0 < -1$ succ. monotona crescente lim. sup da -1

$x_n \nearrow -1$

$-1 < x_0 < 0$ succ. monotona decr. lim. inf da -1

$x_n \searrow -1$

$0 < x_0 < 1$ succ " cresc. " sup da 1

$x_n \nearrow +1$

$1 < x_0$ succ. " decres. lim inf da 1

$x_n \searrow 1$

$\Rightarrow$ $x_0 < 0$ $x_n \to -1$ ordine 3

$x_0 > 0$ $x_n \to 1$

"$g(x_0) = 0$" $\Rightarrow$ succ. costante

$\overset{\shortparallel}{0}$

2) Si consideri la seguente matrice A bidiagonale di ordine N,

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$$A = \begin{pmatrix} 1 & 3 & & & 0 \\ & 1 & 3 & & \\ & & \ddots & \ddots & \\ & & & 1 & 3 \\ 0 & & & & 1 \end{pmatrix}.$$

Calcolare A^{-1} ed i numeri di condizionamento $K_\infty(A)$ e $K_1(A)$.

ES: $N=5$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 9 & -27 & 81 \\ 0 & 1 & -3 & 9 & -27 \\ 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(A^{-1})_{ij} = (-3)^{j-i} \qquad j \geq i$$

$$(A^{-1})_{ij} = 0 \qquad j < i$$

$\|A\|_\infty = 4$

$$\|A^{-1}\|_\infty = \sum_{j=1}^{N} (3)^{j-1} \; * \qquad (1^\wedge \text{ riga})$$

N.B. $$\sum_{j=0}^{N} q^j = \frac{q^{N+1}-1}{q-1}$$

$$\sum_{j=1}^{N} (3)^{j-1} = \frac{1}{3}\sum_{j=1}^{N} 3^j = \frac{1}{3}\left[\sum_{j=0}^{N} 3^j - 1\right] = \frac{1}{3}\left[\frac{3^{N+1}-1}{3-1} - 1\right] =$$

$$\frac{1}{3}\left[\frac{3^{N+1}}{2} - \frac{1}{2} - 1\right] = \frac{1}{3}\cdot\frac{3^N\cdot 3}{2} - \frac{1}{3}\cdot\frac{3}{2} = \frac{3^N}{2} - \frac{1}{2} = \frac{(3^N-1)}{2}$$

$$K_\infty(A) = K_1(A) = 4 \cdot \frac{3^N-1}{2} = 2(3^N-1)$$

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3) Si consideri l'equazione non lineare $f(x) \equiv e^x - ax = 0,\ a > 0$.

3.1) Studiare il numero di radici reali di f al variare di $a > 0$.

3.2) Per i valori di a per i quali la funzione f ha 2 radici reali distinte α e β $(0 < \alpha < 1 < \beta)$, dimostrare che il metodo di Newton converge alla radice α per ogni $x_0 \in [-1, 1]$.

$f(x) = e^x - ax$

$f'(x) = e^x - a$

$\begin{cases} e^x = ax & \text{Intersezione} \\ e^x = a & \text{Tangenza} \end{cases}$

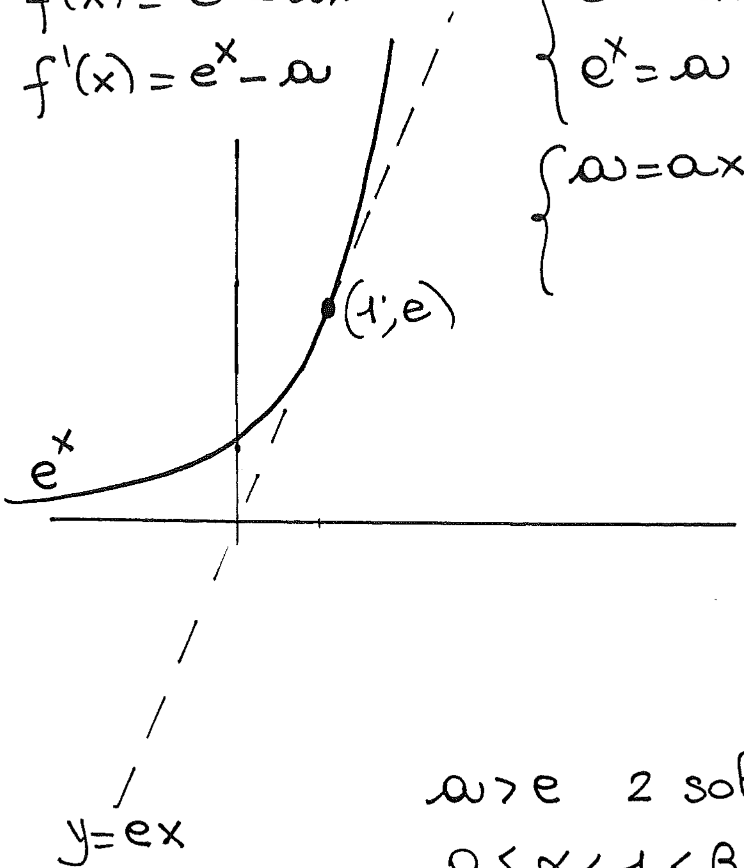
$\begin{cases} a = ax \implies x = 1 \quad P(1; e) \\ a = e \end{cases}$

$\Rightarrow f(x) = e^x - ex$

$f(1) = 0$

$f'(x) = e^x - e$ TANG.

$f'(1) = 0$



$a > e$ 2 soluzioni distinte

$0 < \alpha < 1 < \beta$

$a = e$ 2 soluz. coincidenti

$\alpha = \beta = 1$

$a < e$ nessuna soluzione

caso $\boxed{a > e}$, $I = [-1, 1]$ $\quad f(x) = e^x - ax;\ f'(x) = e^x - a;\ f''(x) = e^x$

1) $f(-1) = e^{-1} + a > 0 \qquad f(1) = e - a < 0 \quad (e < a)$

2) $f'(x) = e^x - a < e - a < 0 \quad (x \in I)$

3) $f''(x) = e^x > 0$

4) $\left|\dfrac{f(-1)}{f'(-1)}\right| = \dfrac{e^{-1} + a}{|e^{-1} - a|} = \dfrac{\frac{1}{e} + a}{a - \frac{1}{e}} = \dfrac{1 + ea}{ae - 1} < 2 \qquad \begin{array}{l} 1 + ea < 2ae - 2 \\ ae > 3 \quad (ae > e^2 > 3) \end{array}$

$\left|\dfrac{f(1)}{f'(1)}\right| = \dfrac{|e - a|}{|e - a|} = 1 < 2$

MdN converge ad $\alpha \in [-1, 1]$ $\forall x_0 \in [-1, 1]$.

4) Dato il sistema lineare $A\mathbf{x} = \mathbf{b}$, con

$$A = \begin{pmatrix} 8 & \beta & 0 \\ \beta & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta \in \mathbb{R}.$$

Siano $C = \{\beta \in \mathbb{R} \mid \exists A^{-1}\}$, $J = \{\beta \in \mathbb{R} \mid \text{il metodo di Jacobi converge}\}$, $GS = \{\beta \in \mathbb{R} \mid \text{il metodo di Gauss} - \text{Seidel converge}\}$.

4.1) Determinare gli insiemi C, J, GS.

4.2) Sia $\beta \in GS$, per quali $\alpha > 0$ il metodo $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \alpha(\mathbf{b} - A\mathbf{x}^{(n)})$ converge?

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4.1)

- $\det A = 32 - \beta^2 \neq 0 \qquad \beta \neq \pm 4\sqrt{2} \qquad C = \mathbb{R} \setminus \{\pm 4\sqrt{2}\}$

- Jacobi

$$\det \begin{bmatrix} 8\lambda & \beta & 0 \\ \beta & 4\lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda\left(32\lambda^2 - \beta^2\right) = 0 \qquad \lambda = 0 \quad \lambda = \pm \frac{\beta}{4\sqrt{2}}$$

$$\rho(B_J) = \frac{|\beta|}{4\sqrt{2}} < 1 \qquad |\beta| < 4\sqrt{2} \qquad J = (-4\sqrt{2}; 4\sqrt{2})$$

- GSeidel

Matrice tridiagonale.... $\rho(B_{GS}) = \frac{\beta^2}{32} < 1 \quad |\beta| < 4\sqrt{2}$

$GS = (-4\sqrt{2}; 4\sqrt{2})$

4.2) $B = I - \alpha A$

Problema autovalori per B : $\mu \in \sigma(B)$

$(I - \alpha A)x = \mu x$

$x - \alpha A x = \mu x \qquad \alpha A x = (1-\mu)x \qquad Ax = \frac{1-\mu}{\alpha} x$

$\frac{1-\mu}{\alpha} = \lambda$ dove λ sono gli autovalori di A

$1 - \mu = \alpha\lambda \qquad \mu = 1 - \alpha\lambda$

Calcolo λ

$$\det(A-\lambda I)=0$$

$$\det\begin{bmatrix} 8-\lambda & \beta & 0 \\ \beta & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}=0$$

$$(1-\lambda)\left[(8-\lambda)(4-\lambda)-\beta^2\right]=0$$

$$(1-\lambda)(32-12\lambda+\lambda^2-\beta^2)=0$$

$$(1-\lambda)(\lambda^2-12\lambda+32-\beta^2)=0$$

$$\lambda=1$$

$$\lambda=6\pm\sqrt{36-32+\beta^2}=6\pm\sqrt{4+\beta^2}\ \begin{cases} 6+\sqrt{4+\beta^2} \\ 6-\sqrt{4+\beta^2} \end{cases}$$

$$\mu_1=1-\alpha$$

$$\mu_2=1-\alpha(6+\sqrt{4+\beta^2})$$

$$\mu_3=1-\alpha(6-\sqrt{4+\beta^2})$$

Convergenza (senza sfruttare risultato)

- $|1-\alpha|<1 \qquad 0<\alpha<2$
- $|1-\alpha(6+\sqrt{4+\beta^2})|<1$

$$\begin{cases} 1-\alpha(6+\sqrt{4+\beta^2})<1 \\ 1-\alpha(6+\sqrt{4+\beta^2})>-1 \end{cases} \qquad \begin{cases} \alpha>0 \\ \alpha<\dfrac{2}{6+\sqrt{4+\beta^2}} \end{cases} \qquad 0<\alpha<\frac{2}{6+\sqrt{4+\beta^2}}$$

Teoria Richardson

$$\mu<\frac{2}{\lambda_{max}}=\frac{2}{6+\sqrt{4+\beta^2}}$$

- $\left|1-\alpha\left(6-\sqrt{4+\beta^2}\right)\right| < 1$

<u>OSS</u>: $6-\sqrt{4+\beta^2} > 0$

$6 > \sqrt{4+\beta^2} \quad 36 > 4+\beta^2$

$32 > \beta^2 \quad (\beta \in GS)$

$$\begin{cases} 1-\alpha\left(6-\sqrt{4+\beta^2}\right) < 1 \\ 1-\alpha\left(6-\sqrt{4+\beta^2}\right) > -1 \end{cases} \qquad \begin{cases} \alpha > 0 \\ \alpha < \dfrac{2}{6-\sqrt{4+\beta^2}} \end{cases}$$

Confronti

$$\frac{2}{6+\sqrt{4+\beta^2}} < 2$$

$$\frac{2}{6+\sqrt{4+\beta^2}} < \frac{2}{6-\sqrt{4+\beta^2}}$$

$\Rightarrow$ convergenza $\qquad 0 < \alpha < \dfrac{2}{6+\sqrt{4+\beta^2}}$