CALCOLO NUMERICO 1 (26 Gennaio 2017) - Prova scritta

COMMENTARE TUTTI I PASSAGGI E GIUSTIFICARE LE RISPOSTE

1) Determinare i parametri $B,\ D,\ b,\ d \in \mathbb{R}$ in modo tale che la funzione

$$S(x) = \begin{cases} 1 + B(x-1) - D(x-1)^3 & 1 \le x < 2 \\ 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3 & 2 \le x \le 3 \end{cases}$$

sia una spline cubica sull'intervallo $[1,3]$ interpolante i dati $(1,1)$, $(2,1)$, $(3,0)$. Dire se la spline trovata è naturale.

$S(1)=1$ vero $\forall\ B, D, b, d$

$S(2)=1$ $\begin{cases} S(2^-) = 1+B-D = 1 \Rightarrow B-D=0 \\ S(2^+) = 1 \quad \forall B, D, b, d \end{cases}$ $\Rightarrow C^0[1,3]$

$S(3)=0$ $\quad 1+b-\frac{3}{4}+d=0 \quad \Rightarrow b+d=-\frac{1}{4}$

$$S'(x) = \begin{cases} B-3D(x-1)^2 & [1,2) \\ b-\frac{3}{2}(x-2)+3d(x-2)^2 & [2,3] \end{cases}$$

$S'(2^-) = B-3D$

$S'(2^+) = b$ $\quad \Rightarrow \quad B-3D=b$

$$S''(x) = \begin{cases} -6D(x-1) \\ -\frac{3}{2}+6d(x-2) \end{cases}$$

$S''(2^-) = -6D$

$S''(2^+) = -\frac{3}{2}$ $\quad \Rightarrow \quad 6D=\frac{3}{2} \quad D=\frac{1}{4}$

$$\begin{cases} B = D \\ b+d = -\frac{1}{4} \\ B-3D = b \\ D = \frac{1}{4} \end{cases} \qquad \begin{cases} B = \frac{1}{4} \\ -\frac{1}{2}+d = -\frac{1}{4} \\ \frac{1}{4}-\frac{3}{4} = b \\ D = \frac{1}{4} \end{cases} \quad \begin{matrix} d = \frac{1}{4} \\ b = -\frac{1}{2} \end{matrix} \qquad \begin{cases} B = \frac{1}{4} \\ D = \frac{1}{4} \\ b = -\frac{1}{2} \\ d = \frac{1}{4} \end{cases}$$

$S''(1)=0 \qquad S''(3) = -\frac{3}{2}+6d = -\frac{3}{2}+6\cdot\frac{1}{4} = 0$

E' NATURALE

2) Assegnato l'intervallo $[a, b]$ e il punto medio $x_M = \frac{a+b}{2}$, trovare ω_0, ω_1 e ω_2 in modo tale che la formula di quadratura,

$$\int_a^b f(x)\mathrm{d}x \simeq \omega_0 f(x_M) + \omega_1 f'(x_M) + \omega_2 f''(x_M)$$

abbia grado di precisione almeno 2.

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- $r=0 \quad f=1 \quad f'=0 \quad f''=0$

$$\int_a^b 1\,dx = (b-a) \qquad \omega_0 = b-a$$

- $r=1 \quad f=x \quad f'=1 \quad f''=0$

$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$$

$$\omega_0 \frac{a+b}{2} + \omega_1 = \frac{b^2-a^2}{2} \qquad \frac{(b-a)(b+a)}{2} + \omega_1 = \frac{b^2-a^2}{2} \Rightarrow \omega_1 = 0$$

- $r=2 \quad f=x^2 \quad f'=2x \quad f''=2$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3-a^3}{3} = \frac{(b-a)(b^2+ab+b^2)}{3}$$

$$(b-a)\left(\frac{a+b}{2}\right)^2 + \omega_2 \cdot 2 = \frac{(b-a)(b^2+ab+b^2)}{3}$$

$$\omega_2 = \frac{1}{2}\left\{(b-a)\left[\frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}\right]\right\}$$

$$= \frac{1}{2}(b-a)\,\frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} =$$

$$\frac{1}{24}(b-a)(b^2-2ab+a^2) = \frac{1}{24}(b-a)^3$$

3) Discutere la convergenza e l'ordine della famiglia di metodi iterativi definiti in funzione del parametro reale a:

$$x_{k+1} = \frac{x_k}{3}(4 - a^3 x_k^3), \ k \geq 0,$$

al variare di $x_0 \in \mathbb{R}$.

- $\boxed{a = 0}$ $\quad x_{k+1} = \frac{4}{3} x_k \quad g'(x) = \frac{4}{3} > 1$ Diverge $\forall x_0$

  P.F. $\alpha = 0$

- $\boxed{a \neq 0}$

- $g(x) = \frac{4}{3}x - \frac{1}{3}a^3 x^4 \qquad g(x) = 0 \quad x = 0 \vee x = \frac{\sqrt[3]{4}}{a}$

$g(x) = x \qquad x = \frac{4}{3}x - \frac{1}{3}a^3 x^4$

$\frac{1}{3}a^3 x^4 - \frac{1}{3}x = 0 \qquad x = 0$

$x = \frac{1}{a}$

$g'(x) = -\frac{4}{3}a^3 x^3 + \frac{4}{3} \qquad g'(0) = \frac{4}{3} > 1$ DIV LOCALE

$g'\left(\frac{1}{a}\right) = -\frac{4}{3} + \frac{4}{3} = 0$ CONV LOC $\boxed{(\text{ord} \geq 2)}$

$g''(x) = -4a^3 x^2 \qquad g''\left(\frac{1}{a}\right) \neq 0 \Rightarrow \boxed{\Rightarrow \text{ord. } 2}$

$\boxed{a > 0}$ $\quad \frac{1}{a} > 0$

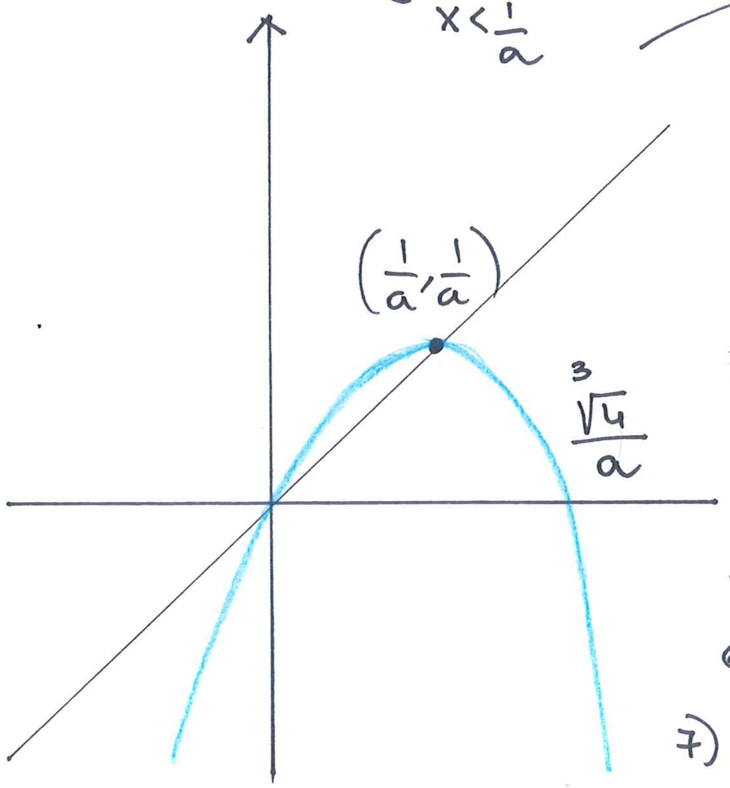
$(0) \quad \frac{1}{a}$

$+ \quad -$

$g'(x) > 0$

$x < \frac{1}{a}$

max: $\left(\frac{1}{a}; \frac{1}{a}\right)$



1) $x_0 < 0$ succ. mon. decr. illim. inf $x_n \searrow -\infty$

2) $x_0 = 0$ "$x_n = 0$" $\forall n$

3) $0 < x_0 < \frac{1}{a}$ succ. mon cresc. lim sup. $x_n \nearrow \frac{1}{a}$

4) $x_0 = \frac{1}{a}$ "$x_n = \frac{1}{a}$" $\forall n$

5) $\frac{1}{a} < x_0 < \frac{\sqrt[3]{4}}{a}$ $\quad 0 < x_1 < \frac{1}{a}$ caso 3)

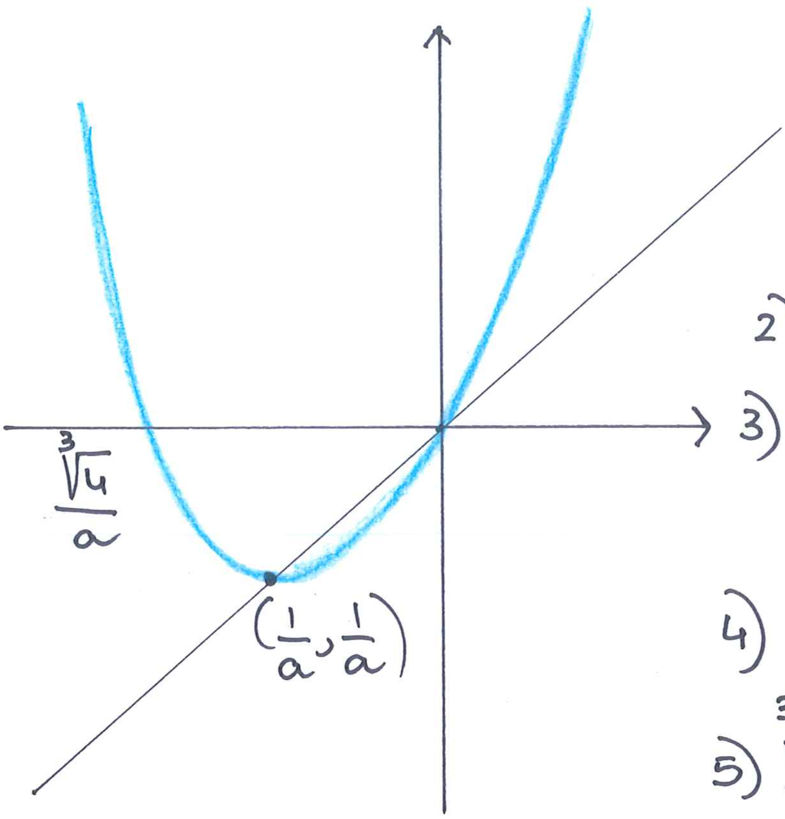
6) $x_0 = \frac{\sqrt[3]{4}}{a}$ $\quad x_1 = 0$ caso 2)

7) $x_0 > \frac{\sqrt[3]{4}}{a}$ $\quad x_1 < 0$ caso 1)

$\boxed{a < 0}$ $\quad \frac{1}{a} < 0$ $\quad 1/a \quad (0)$

$g'(x) > 0 \quad x > \frac{1}{a}$ $\quad - \quad +$

min : $\left(\frac{1}{a}, \frac{1}{a}\right)$



1) $x_0 > 0$ succ. mon. cresc ill. sup. $x_n \nearrow +\infty$

2) $x_0 = 0$ "$x_n = 0$" $\forall n$

3) $\frac{1}{a} < x_0 < 0$ succ. mon. decr. lim. inf. $x_n \searrow \frac{1}{a}$

4) $x_0 = \frac{1}{a}$ $\quad x_n = \frac{1}{a}$ $\quad \forall n$

5) $\frac{\sqrt[3]{4}}{a} < x_0 < \frac{1}{a}, \frac{1}{a} < x_1 < 0$ caso 3)

6) $x_0 = \frac{\sqrt[3]{4}}{a}$ $\quad x_1 = 0$ caso 2)

7) $x_0 < \frac{\sqrt[3]{4}}{a}$ $\quad x_1 > 0$ caso 1)

RIEPILOGO

$a > 0 \quad 0 < x_0 < \frac{\sqrt[3]{4}}{a}$ $\quad x_n \to \frac{1}{a}$ $\quad$ ordine 2

$a < 0 \quad \frac{\sqrt[3]{4}}{a} < x_0 < 0$ $\quad x_n \to \frac{1}{a}$ $\quad$ ordine 2

4) Dato un sistema lineare con matrice

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}, \quad a \in \mathbb{R}\backslash\{0\}:$$

4.1) trovare per quali valori di a la matrice è definita positiva;

4.2) trovare per quali valori di a il metodo di Jacobi è convergente;

4.2) trovare l'espressione del numero di condizionamento di $K_2(A)$ in funzione di a.

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4.1 $\det A_1 = 1$

$\det A_2 = 1 - a^2 > 0 \qquad |a| < 1$

$\det A_3 = 1(1-a^2) - a(a-a^2) + a(a^2-a) =$

$= 1 - a^2 - a^2 + a^3 + a^3 - a^2 = 2a^3 - 3a^2 + 1$

$p(1) = 0$

$$\begin{array}{c|ccc|c} & 2 & -3 & 0 & 1 \\ 1 & & 2 & -1 & -1 \\ \hline & 2 & -1 & -1 & // \\ 1 & & 2 & 1 & \\ \hline & 2 & 1 & // & \end{array}$$

$(a-1)^2(2a+1) > 0$

$a > -\frac{1}{2} \wedge a \neq 1$

A def pos $\Longleftrightarrow -\frac{1}{2} < a < 1$

4.2. $\det(\lambda D + L_A + U_A) = 0 \Longleftrightarrow \det(B_J - \lambda I) = 0$

$$\det \begin{vmatrix} \lambda & a & a \\ a & \lambda & a \\ a & a & \lambda \end{vmatrix} = \lambda(\lambda^2 - a^2) - a(a\lambda - a^2) + a(a^2 - a\lambda) =$$

$= \lambda^3 - a^2\lambda - a^2\lambda + a^3 + a^3 - a^2\lambda =$

$= \lambda^3 - 3a^2\lambda + 2a^3 = 0$

$$\begin{array}{c|ccc|c} & 1 & 0 & -3a^2 & 2a^3 \\ a & & a & a^2 & -2a^3 \\ \hline & 1 & a & -2a^2 & / \\ a & & a & 2a^2 & \\ \hline & 1 & 2a & // & \end{array}$$

$(\lambda - a)^2(\lambda + 2a) = 0$

$\lambda = a$ molt. 2

$\lambda = -2a$

$\rho(B) = 2|a| < 1 \qquad |a| < \frac{1}{2}$

$$K_2(a) = \frac{\max|\lambda(A)|}{\min|\lambda(A)|} \qquad (A \text{ è simmetrica})$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{bmatrix} 1-\lambda & a & a \\ a & 1-\lambda & a \\ a & a & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(1-\lambda)^2 - a^2\right] - a\left[a(1-\lambda) - a^2\right] + a\left[a^2 - a(1-\lambda)\right] =$$

$$(1-\lambda)^3 - a^2(1-\lambda) - a^2(1-\lambda) + a^3 + a^3 - a^2(1-\lambda) =$$

$$(1-\lambda)^3 - 3a^2(1-\lambda) + 2a^3 = 0 \qquad (1-\lambda) = t$$

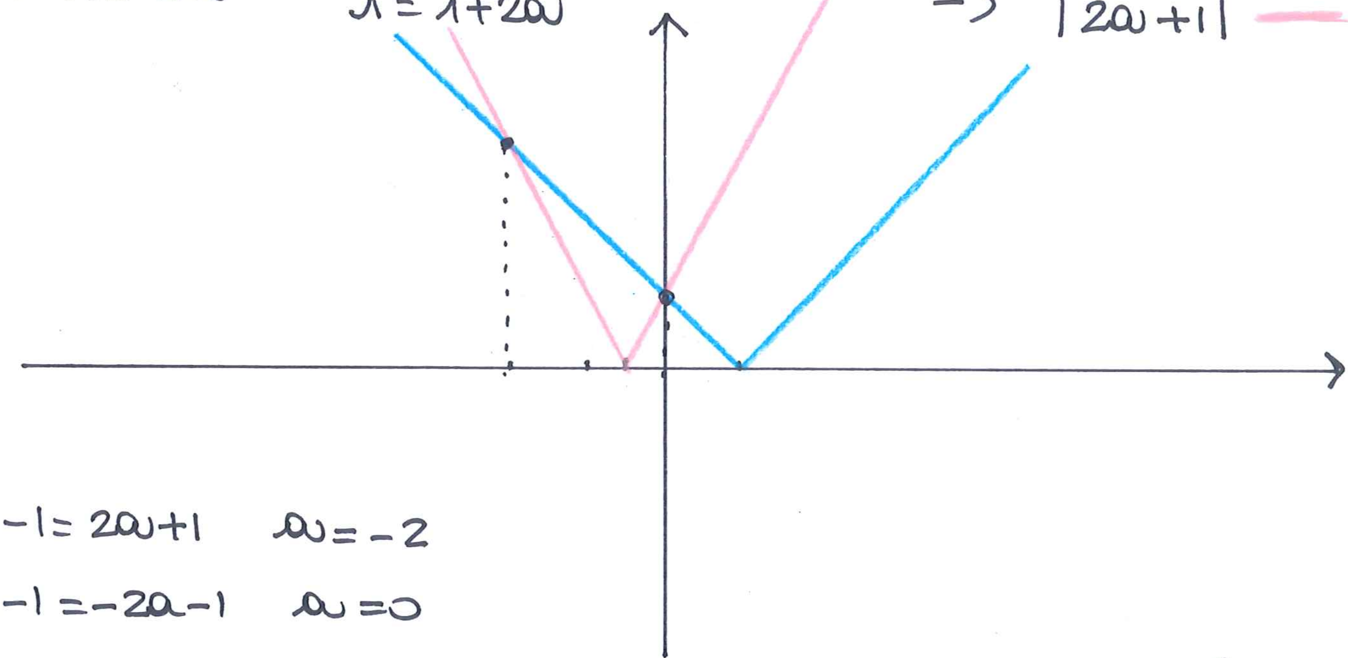
$$t^3 - 3a^2 t + 2a^3 = 0 \qquad (\text{cfr } \lambda(B_J))$$

$1-\lambda = a$ $\quad \lambda = 1-a$ $\quad$ molt. 2.

$1-\lambda = -2a$ $\quad \lambda = 1+2a$ $\quad \Rightarrow$

$|a-1|$

$|2a+1|$



$a-1 = 2a+1 \qquad a = -2$

$a-1 = -2a-1 \qquad a = 0$

$$K_2(A) = \begin{cases} \dfrac{|2a+1|}{|a-1|} & a \leq -2 \ \vee \ a \geq 0 \quad \left(\begin{matrix} A = I \\ \text{se } a = 0 \end{matrix}\right) \\[2ex] \dfrac{|a-1|}{|2a+1|} & -2 < a < 0 \end{cases}$$

5) Dato $h > 0$, indicare per quali valori di $\beta \in [0, 2h]$ esiste uno ed unico polinomio di terzo grado p_3 tale che

$$p_3(0) = f(0), \quad p_3(h) = f(h), \quad p_3(2h) = f(2h), \quad p_3'(\beta) = f'(\beta),$$

per ogni $f \in C^1([0, 2h])$.

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$$p_3(x) = ax^3 + bx^2 + cx + d$$

$$p_3'(x) = 3ax^2 + 2bx + c$$

$$p_3(0) = d = f(0)$$

$$p_3(h) = ah^3 + bh^2 + ch + d = f(h)$$

$$p_3(2h) = 8ah^3 + 4bh^2 + 2ch + d = f(2h)$$

$$p_3'(\beta) = 3a\beta^2 + 2b\beta + c = f'(\beta)$$

$$\begin{bmatrix} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \\ h^3 & h^2 & h & 1 \\ 8h^3 & 4h^2 & 2h & 1 \\ 3\beta^2 & 2\beta & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} f(0) \\ f(h) \\ f(2h) \\ f'(\beta) \end{bmatrix}$$

$$\begin{vmatrix} h^3 & h^2 & h \\ 8h^3 & 4h^2 & 2h \\ 3\beta^2 & 2\beta & 1 \end{vmatrix} = h^3(4h^2 - 4\beta h) - 8h^3(h^2 - 2\beta h) + 3\beta^2(2h^3 - 4h^3) =$$

$$4h^5 - 4\beta h^4 - 8h^5 + 16\beta h^4 - 6\beta^2 h^3 =$$

$$-4h^5 + 12\beta h^4 - 6\beta^2 h^3 = 0$$

$$h^3 \left[4h^2 - 12\beta h + 6\beta^2\right] = 0 \qquad 3\beta^2 - 6\beta h + 2h^2 = 0$$

$$\beta = \frac{3h \pm \sqrt{9h^2 - 6h^2}}{3} = \frac{3h \pm h\sqrt{3}}{3} = h\left(1 \pm \frac{\sqrt{3}}{3}\right)$$

$$\beta \neq \left(1 \pm \frac{\sqrt{3}}{3}\right) h$$