1) Dato il sistema lineare $A\mathbf{x} = \mathbf{b}$, con

$$A = \begin{pmatrix} a & 0 & 1/a & 0 \\ 0 & a & 0 & 1/a \\ 1/a & 0 & a & 0 \\ 0 & 1/a & 0 & a \end{pmatrix}, \ a \neq 0, \quad \mathbf{b} \in \mathbb{R}^{4,1},$$

trovare una condizione necessaria e sufficiente su a per la convergenza dei metodi iterativi di Jacobi e di Gauss Seidel e confrontare le rispettive velocità di convergenza.

Discutere la convergenza del metodo iterativo

$$M\mathbf{x}^{(n+1)} = N\mathbf{x}^{(n)} + \mathbf{b}, \quad n \geq 0, \quad \mathbf{x}^{(0)} \text{ dato}, \ M = \begin{pmatrix} a/2 & 0 & 0 & 0 \\ 0 & a/2 & 0 & 0 \\ 0 & 0 & a/2 & 0 \\ 0 & 0 & 0 & a/2 \end{pmatrix}.$$

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Metodo di Jacobi

$$\det \begin{bmatrix} a\lambda & 0 & \frac{1}{a} & 0 \\ 0 & a\lambda & 0 & \frac{1}{a} \\ \frac{1}{a} & 0 & a\lambda & 0 \\ 0 & \frac{1}{a} & 0 & a\lambda \end{bmatrix} = a\lambda \begin{vmatrix} a\lambda & 0 & \frac{1}{a} \\ 0 & a\lambda & 0 \\ \frac{1}{a} & 0 & a\lambda \end{vmatrix} + \frac{1}{a} \begin{vmatrix} 0 & \frac{1}{a} & 0 \\ a\lambda & 0 & \frac{1}{a} \\ \frac{1}{a} & 0 & a\lambda \end{vmatrix} =$$

$$= a\lambda \cdot a\lambda\left[(a\lambda)^2 - \frac{1}{a^2}\right] - \left(\frac{1}{a}\right)^2\left[(a\lambda)^2 - \frac{1}{a^2}\right] = \left(a^2\lambda^2 - \frac{1}{a^2}\right)^2 = 0$$

$$\lambda^2 = \frac{1}{a^4} \qquad \lambda = \pm\frac{1}{a^2} \qquad \rho(B_J) = \frac{1}{a^2} < 1 \iff |a| > 1$$

Metodo di Gauss-Seidel

$$\det \begin{bmatrix} a\lambda & 0 & \frac{1}{a} & 0 \\ 0 & a\lambda & 0 & \frac{1}{a} \\ \frac{1}{a}\lambda & 0 & a\lambda & 0 \\ 0 & \frac{1}{a}\lambda & 0 & a\lambda \end{bmatrix} = a\lambda \begin{bmatrix} a\lambda & 0 & \frac{1}{a} \\ 0 & a\lambda & 0 \\ \frac{1}{a}\lambda & 0 & a\lambda \end{bmatrix} + \frac{1}{a}\lambda \begin{bmatrix} 0 & \frac{1}{a} & 0 \\ a\lambda & 0 & \frac{1}{a} \\ \frac{1}{a}\lambda & 0 & a\lambda \end{bmatrix}$$

$$= (a\lambda)^2\left(a^2\lambda^2 - \frac{1}{a^2}\lambda\right) - \frac{1}{a^2}\lambda\left(a^2\lambda^2 - \frac{1}{a^2}\lambda\right) = 0$$

$$\left(a^2\lambda^2 - \frac{1}{a^2}\lambda\right)^2 = 0 \qquad \lambda = 0 \quad \lambda = \frac{1}{a^4} \qquad \rho(B_{GS}) = \frac{1}{a^4} < 1 \iff |a| > 1$$

$$R(B_{GS}) = -\ln\frac{1}{a^4} = 2\left(-\ln\frac{1}{a^2}\right) = 2R\,B_J \qquad \text{(velocità doppia)}$$

$$Mx^{(n+1)} = Nx^{(n)} + b \qquad A = M-N \ , \ N = M-A$$

$$\downarrow n\to\infty$$

$$Mx = Nx + b \qquad B = M^{-1}N \quad \text{matrice di iterazione}$$

$$B = \begin{bmatrix} \frac{2}{a} & 0 & 0 & 0 \\ 0 & \frac{2}{a} & 0 & 0 \\ 0 & 0 & \frac{2}{a} & 0 \\ 0 & 0 & 0 & \frac{2}{a} \end{bmatrix} \begin{bmatrix} -\frac{a}{2} & 0 & -\frac{1}{a} & 0 \\ 0 & -\frac{a}{2} & 0 & -\frac{1}{a} \\ -\frac{1}{a} & 0 & -\frac{a}{2} & 0 \\ 0 & -\frac{1}{a} & 0 & -\frac{a}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\frac{2}{a^2} & 0 \\ 0 & -1 & 0 & -\frac{2}{a^2} \\ -\frac{2}{a^2} & 0 & -1 & 0 \\ 0 & -\frac{2}{a^2} & 0 & -1 \end{bmatrix}$$

$$\lambda(B) = -\lambda(\underbrace{-B}_{C}) \qquad \Rightarrow \det(C - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 0 & \frac{2}{a^2} & 0 \\ 0 & 1-\lambda & 0 & +\frac{2}{a^2} \\ +\frac{2}{a^2} & 0 & 1-\lambda & 0 \\ 0 & +\frac{2}{a^2} & 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 0 & +\frac{2}{a^2} \\ 0 & 1-\lambda & 0 \\ \frac{2}{a^2} & 0 & 1-\lambda \end{vmatrix} + \frac{2}{a^2} \begin{vmatrix} 0 & \frac{2}{a^2} & 0 \\ 1-\lambda & 0 & \frac{2}{a^2} \\ \frac{2}{a^2} & 0 & 1-\lambda \end{vmatrix} =$$

$$(1-\lambda)^2 \left[(1-\lambda)^2 - \frac{4}{a^4}\right] - \frac{4}{a^4}\left[(1-\lambda)^2 - \frac{4}{a^4}\right] = 0$$

$$\left[(1-\lambda)^2 - \frac{4}{a^4}\right]^2 = 0 \qquad (1-\lambda)^2 = \frac{4}{a^4}$$

$$\left.\begin{aligned} &1-\lambda = \frac{2}{a^2} \quad \vee \quad 1-\lambda = -\frac{2}{a^2} \\ &\lambda_{1,2} = 1 - \frac{2}{a^2} \quad \vee \quad \lambda_{3,4} = 1 + \frac{2}{a^2} \end{aligned}\right\} \begin{aligned} &\lambda_{3,4} > 1 \\ &\Rightarrow \rho(B) = \rho(C) > 1 \end{aligned}$$

2) Assegnati i punti $(0,-1)$, $(1,0)$, $(3,-1)$:

2.1) costruire la spline lineare interpolante;

2.2) costruire la generica spline quadratica interpolante;

2.3) determinare se esiste una spline quadratica interpolante $s_2(x)$ tale che $s_2'(0) = 1$.

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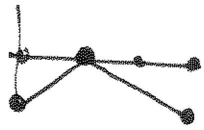
2.1) Spline lineare interpolante

x_i	0	1	3
y_i	-1	0	-1



$$S_1(x) = \begin{cases} x-1 & x \in [0,1) \\ -\frac{1}{2}x + \frac{1}{2} & x \in [1,3] \end{cases}$$

2.2) $$S_2(x) = \begin{cases} ax^2 + bx + c & x \in [0,1) \\ d(x-1)^2 + e(x-1) + f & x \in [1,3] \end{cases}$$

$$S_2'(x) = \begin{cases} 2ax + b & — \\ 2d(x-1) + e & — \end{cases}$$

$S_2(0) = -1 \qquad c = -1$

$S_2(1^-) = a + b + c = 0$

$S_2(1^+) = f = 0$

$S_2(3) = 4d + 2e + f = -1$

$S_2'(1^-) = S_2'(1^+) \qquad 2a + b = e$

$$\begin{cases} c = -1 \\ f = 0 \\ a + b = 1 \\ 4d + 2e = -1 \\ 2a + b = e \end{cases} \qquad \begin{cases} a = -b + 1 \\ 4d + 2e = -1 \\ -2b + 2 + b = e \end{cases} \qquad \begin{cases} a = -b + 1 \\ 4d - 2b = -1 \\ e = -b + 2 \end{cases} \qquad \begin{cases} a = -b + 1 \\ d = \frac{2b-5}{4} \\ e = -b + 2 \end{cases}$$

$$S_2(x) = \begin{cases} (-b+1)x^2 + bx - 1 & x \in [0,1) \\ \frac{2b-5}{4}(x-1)^2 + (-b+2)(x-1) & x \in [1,3] \end{cases} \qquad b \in \mathbb{R}$$

Condizione aggiuntiva

$S'(0) = 1$

$S'(x) = 2ax + b \qquad x \in [0,1)$

$= 2(-b+1)x + b$

$S'(0) = 1 \qquad b = 1$

$$S_2(x) = \begin{cases} x - 1 & x \in [0,1) \\ -\frac{3}{4}(x-1)^2 + (x-1) & x \in [1,3] \end{cases}$$

3) Dati $x_0 = -\pi$, $x_1 = 0$, $x_2 = \pi$:

3.1) costruire il polinomio $p(x)$ interpolante $f(x) = x^2 + sin(x)$ nei nodi x_i, $i = 0, 1, 2$;

3.2) costruire polinomio $q(x)$ interpolante $g(x) = x^3 + sin(x)$ nei nodi x_i, $i = 0, 1, 2$;

3.3) trovare una maggiorazione degli errori

$$\max_{x\in[-\pi,\pi]} |f(x) - p(x)|, \quad \max_{x\in[-\pi,\pi]} |g(x) - q(x)|.$$

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$f(x)$:

x_i	$-\pi$	0	π
$f(x_i)$	π^2	0	π^2

$p(x) = x^2$

$n=2$

$n+1=3$ punti

$g(x)$:

x_i	$-\pi$	0	π
$g(x_i)$	$-\pi^3$	0	π^3

$q(x) = \pi^2 x$

Per la maggiorazione dell'errore:

$\omega(x) = (x+\pi)\,x\,(x-\pi) = x^3 - \pi^2 x$

$\max_{x\in[-\pi,\pi]} |\omega(x)|$: $\quad \omega'(x) = 3x^2 - \pi^2 > 0 \quad x < -\frac{\pi}{\sqrt{3}} \cup x > \frac{\pi}{\sqrt{3}}$

$$\omega\left(\frac{\pi}{\sqrt{3}}\right) = \frac{\pi^3}{3\sqrt{3}} - \frac{\pi^3}{\sqrt{3}} = -\frac{2}{3\sqrt{3}}\pi^3$$

$$\omega\left(-\frac{\pi}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}\pi^3$$

$$\max_{x\in[-\pi,\pi]} |\omega(x)| = \frac{2\sqrt{3}}{9}\pi^3$$

$$\max_{-\pi\le x\le\pi} |f(x) - p(x)| \le \frac{1}{6}\cdot\frac{2\sqrt{3}}{9}\pi^3 \max_{t\in[-\pi,\pi]} |f'''(t)| \le \frac{\sqrt{3}}{27}\pi^3$$

$f'(t) = 2t + \cos t$

$f''(t) = 2 - \operatorname{sen} t$

$f'''(t) = -\cos t$

$$\max_{-\pi\le x\le\pi} |g(x) - q(x)| \le \frac{1}{6}\cdot\frac{2\sqrt{3}}{9}\pi^3 \max_{-\pi\le t\le\pi} |g'''(t)| \le \frac{\sqrt{3}}{27}\pi^3\cdot 7$$

$$\left.\begin{aligned} g'(t) &= 3t^2 + \cos t \\ g''(t) &= 6t - \operatorname{sen} t \\ g'''(t) &= 6 - \cos t \end{aligned}\right\} 5 \le |g'''(t)| \le 7$$

4) Data la formula di quadratura con peso $|x|$:

$$\int_{-1}^{1} |x|\, f(x)dx \approx \alpha f(x_0) + \beta f(x_1), \quad x_0 \neq x_1,\ x_0,\ x_1 \neq 0$$

trovare i pesi α, β e i nodi x_0, x_1, in modo che abbia grado di precisione massimo. Determinare il grado di precisione della formula ottenuta.

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$f=1 \quad \int_{-1}^{1} |x|\, dx = 1 \qquad \alpha+\beta=1$

$f=x \quad \int_{-1}^{1} |x|\, x\, dx = 0 \qquad \alpha(+x_0)+\beta(x_1)=0$

$f=x^2 \quad \int_{-1}^{1} |x|\, x^2 dx = 2\int_0^1 x^3 dx = \frac{1}{2} \qquad \alpha \cdot x_0^2 + \beta x_1^2 = \frac{1}{2}$

$f=x^3 \quad \int_{-1}^{1} |x|\, x^3 dx = 0 \qquad \alpha x_0^3 + \beta x_1^3 = 0$

$$\begin{cases} \alpha+\beta=1 \\ +\alpha x_0 + \beta x_1 = 0 \\ \alpha x_0^2 + \beta x_1^2 = \frac{1}{2} \\ \alpha x_0^3 + \beta x_1^3 = 0 \end{cases}$$

2^a: $x_0 = -\frac{\beta}{\alpha} x_1$

4^a: $x_0^3 = -\frac{\beta x_1^3}{\alpha} \quad -\frac{\beta^3}{\alpha^3} x_1^3 = -\frac{\beta}{\alpha} x_1^3 \quad \alpha = \pm\beta$

($\alpha = -\beta$ incompatibile con 1^a equazione)

$$\begin{cases} 2\alpha = 1 \quad \alpha=\beta=\frac{1}{2} \\ 2^a: x_0 = -x_1 \\ \alpha x_0^2 + \alpha x_0^2 = \frac{1}{2} \\ \dots \end{cases} \qquad \begin{cases} 2x_0^2 = 1 \end{cases} \quad x_0 = -\frac{1}{\sqrt{2}} \quad x_1 = \frac{1}{\sqrt{2}} \dots$$

Grado di precisione: almeno 3

$f = x^4 \quad \int_{-1}^{1} |x|\, x^4 dx = 2\int_0^1 x^5 dx = 2 \cdot \frac{x^6}{6}\Big|_0^1 = \frac{1}{3}$

F.Q. $\frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)^4 + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4}$ $\Rightarrow$ G.P. 3

(Formula a 2 nodi $\Rightarrow$ G.P. $\leq 2 \cdot 2 - 1 = 3$; vedi formule gaussiane)

5) Sia $f : [a,b] \to \mathbb{R}$, $-\infty < a < b < \infty$, una funzione continua, trovare $c \in \mathbb{R}$ in modo tale che sia minima la quantità

$$\sqrt{\int_a^b (f(x) - c)^2 \, dx}$$

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$$\varphi(c) = \sqrt{\int_a^b (f(x) - c)^2 \, dx}$$

Trovare $\min_{c \in \mathbb{R}} \varphi(c)$. Sia c^* t.c. $\varphi(c^*) = \min_{c \in \mathbb{R}} \varphi(c)$

$\psi(c) = [\varphi(c)]^2$. $\psi(c^*) = \min_{c \in \mathbb{R}} \psi(c)$

$$\psi(c) = \int_a^b [f(x)]^2 dx - 2 \int_a^b f(x) c \, dx + \int_a^b c^2 dx$$

$$\psi'(c) = -2 \int_a^b f(x) dx + 2c(b-a) > 0$$

$$c < \frac{1}{b-a} \int_a^b f(x) \, dx$$

$$c^* = \frac{1}{b-a} \int_a^b f(x) \, dx$$