CALCOLO NUMERICO 1 (17 novembre 2011) - Prima prova in itinere

1) Studiare il condizionamento $K_f(x)$ della funzione $y = \log(x+1)^2$ e stabilire per quali x il calcolo della funzione è ben condizionato, nel senso che $K_f(x) < 10$.

$$y = \log(x+1)^2 \qquad \text{C.E. } x \neq 1$$

$$y = 2\log|x+1|$$

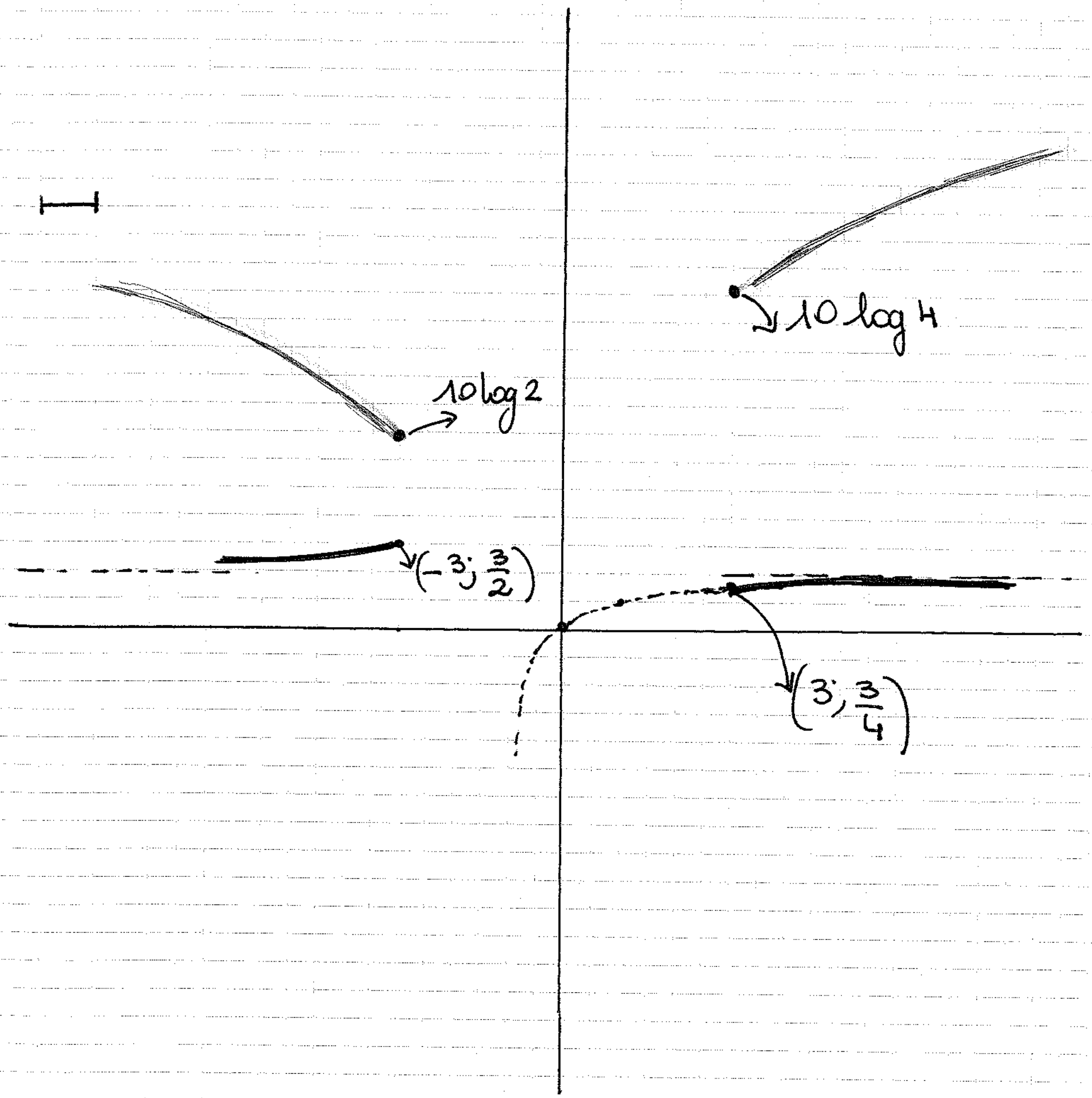
$$y = \begin{cases} 2\log(x+1) & x > -1 \\ 2\log(-x-1) & x < -1 \end{cases}$$

$$y' = \begin{cases} \dfrac{2}{x+1} & x > -1 \\ \dfrac{2}{x+1} & x < -1 \end{cases} = \frac{2}{x+1}$$

$$K_f(x) = \left| \frac{\frac{2x}{x+1}}{2\log|x+1|} \right| \qquad |x+1| \neq 1 \quad \begin{matrix} x \neq 0 \\ x \neq -2 \end{matrix}$$

$$\frac{\left|\frac{x}{x+1}\right|}{|\log|x+1||} < 10$$

$$\left|\frac{x}{x+1}\right| < 10\,|\log|x+1||$$



$\left|\frac{x}{x+1}\right| = \frac{x}{x+1}$ se $|x| > 3$ $\qquad \left|\frac{x}{x+1}\right| \xrightarrow[|x|\to\infty]{} 1$

$\left|\log|x+1|\right| = \log|x+1|$ se $|x|>3$

$10\log|x+1|$

$x=3 \quad 10\log 4$ $\qquad 10\log|x+1| \xrightarrow[|x|\to\infty]{} +\infty$

$x=-3 \quad 10\log 2$

RISPOSTA: SÌ, BEN COND.

I° itinere 17/11/2011

2) Data la funzione $\phi(x) = 3\sin(2\pi x)$:

2.1) calcolare il polinomio p_2 che interpola ϕ nei nodi $x_0 = 0$, $x_1 = \frac{1}{4}$ e $x_2 = \frac{3}{4}$ e il polinomio p_3 che interpola ϕ nei nodi x_0, x_1, x_2 e $x_3 = \frac{1}{2}$;

2.2) fornire una maggiorazione dell'errore $\max_{x\in[0,1]} |\phi(x) - p_2(x)|$;

2.3) calcolare la formula di quadratura interpolatoria relativa ai nodi x_0, x_1, x_2 per l'approssimazione dell'integrale definito $\int_0^1 f(x)dx$. Qual è il suo grado di precisione?

2.1)

x_i	0	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
y_i	0	3	-3	0

$y = 3\sin(2\pi x)$

$$\begin{array}{cccccc}
0 & \boxed{0} & & & & \\
 & & \boxed{12} & & & \\
\frac{1}{4} & 3 & & \frac{-24}{\frac{3}{4}} = \boxed{-32} & & \\
 & & -12 & & \frac{32}{\frac{1}{2}} = \boxed{64} & \\
\frac{3}{4} & -3 & & 0 & & \\
 & & -12 & & & \\
\frac{1}{2} & 0 & & & &
\end{array}$$

$p_2(x) = 12x - 32x\left(x - \frac{1}{4}\right) = -32x^2 + 20x$

$p_3(x) = p_2(x) + 64x\left(x - \frac{1}{4}\right)\left(x - \frac{3}{4}\right)$

$= 32x(2x^2 - 3x + 1)$

VERIFICHE:

$p_2(0) = 0 \quad p_2\left(\frac{1}{4}\right) = -\cancel{32}^{2} \cdot \frac{1}{\cancel{16}} + \cancel{20}^{5} \cdot \frac{1}{\cancel{4}} = 3 \quad p_2\left(\frac{3}{4}\right) = -\cancel{32}^{2} \cdot \frac{9}{\cancel{16}} + \cancel{20}^{5} \cdot \frac{3}{\cancel{4}} = -3$

$p_3(0) = 0 \quad p_3\left(\frac{1}{4}\right) = \cancel{32}^{8} \cdot \frac{1}{\cancel{4}}\left(\cancel{2} \cdot \frac{1}{\cancel{16}^{8}} - 3 \cdot \frac{1}{4} + 1\right) = \cancel{8}\left(\frac{1 - 6 + 8}{\cancel{8}}\right) = 3$

$p_3\left(\frac{3}{4}\right) = \cancel{32}^{8} \cdot \frac{3}{\cancel{4}}\left(\cancel{2} \cdot \frac{9}{\cancel{16}^{8}} - 3 \cdot \frac{3}{4} + 1\right) = \cancel{24}^{3}\left(\frac{9 - 18 + 8}{\cancel{8}}\right) = -3$

$p_3\left(\frac{1}{2}\right) = \cancel{32}^{16} \cdot \frac{1}{\cancel{2}}\left(2 \cdot \frac{1}{4} - 3 \cdot \frac{1}{2} + 1\right) = 16 \cdot 0 = 0$

2.2) $|p_2(x) - \phi(x)| = \frac{1}{3!} |\omega(x)| \, |\phi^{(3)}(t)|$

$$\max_{0 \le x \le 1} |p_2(x) - \phi(x)| \le \frac{1}{6} \max_{0 \le x \le 1} |\omega(x)| \max_{0 \le t \le 1} |\phi^{(3)}(t)|$$

$$\omega(x) = x\left(x - \frac{1}{4}\right)\left(x - \frac{3}{4}\right)$$

A) $|\omega(x)| = |x|\,\left|x - \frac{1}{4}\right| \cdot \left|x - \frac{3}{4}\right| \le 1 \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16} \qquad x \in [0,1]$

B) $\omega(x) = x\left(x^2 - x + \frac{3}{16}\right) = x^3 - x^2 + \frac{3}{16}x$

$$\omega'(x) = 3x^2 - 2x + \frac{3}{16} \ge 0$$

$$48x^2 - 32x + 3 \ge 0 \qquad x_{1,2} = \frac{16 \pm \sqrt{256 - 144}}{48} =$$

$$\frac{16 \pm \sqrt{112}}{48} = \frac{16 \pm 4\sqrt{7}}{48} = \frac{1}{3} \pm \frac{\sqrt{7}}{12}$$

$0 \qquad \frac{1}{3} - \frac{\sqrt{7}}{12} \qquad \frac{1}{3} + \frac{\sqrt{7}}{12} \qquad 1$

$\omega(0) = 0$ (ovvio)

$$\omega\left(\frac{1}{3} - \frac{\sqrt{7}}{12}\right) = \left(\frac{1}{3} - \frac{\sqrt{7}}{12}\right)\left(\frac{1}{3} - \frac{\sqrt{7}}{12} - \frac{1}{4}\right)\left(\frac{1}{3} - \frac{\sqrt{7}}{12} - \frac{3}{4}\right) =$$

$$= \left(\frac{1}{12}\right)^3 (4 - \sqrt{7})(1 - \sqrt{7})(-\sqrt{7} - 5) =$$

$$= \frac{1}{1728}(4 - \sqrt{7} - 4\sqrt{7} + 7)(-\sqrt{7} - 5) =$$

$$= \frac{1}{1728}(11 - 5\sqrt{7})(-\sqrt{7} - 5) = \frac{1}{1728}(-11\sqrt{7} + 35 - 55 + 25\sqrt{7})$$

$$= \frac{1}{1728}(14\sqrt{7} - 20) \approx 0.00986$$

$$\omega\left(\frac{1}{3}+\frac{\sqrt{7}}{12}\right)=\left(\frac{1}{3}+\frac{\sqrt{7}}{12}\right)\left(\frac{1}{3}+\frac{\sqrt{7}}{12}-\frac{1}{4}\right)\left(\frac{1}{3}+\frac{\sqrt{7}}{12}-\frac{3}{4}\right)=$$

$$\left(\frac{1}{12}\right)^3\left(4+\sqrt{7}\right)\left(1+\sqrt{7}\right)\left(-5+\sqrt{7}\right)=$$

$$\frac{1}{1728}\left(4+\sqrt{7}+4\sqrt{7}+7\right)\left(-5+\sqrt{7}\right)=$$

$$\frac{1}{1728}\left(11+5\sqrt{7}\right)\left(-5+\sqrt{7}\right)=$$

$$\frac{1}{1728}\left(-55-25\sqrt{7}+11\sqrt{7}+35\right)=$$

$$\frac{1}{1728}\left(-20-14\sqrt{7}\right)\approx 0.033$$

$$\omega(1)=1\cdot\frac{3}{4}\cdot\frac{1}{4}=\frac{3}{16}=0.1875$$

$$\max_{0\leq x\leq 1}|\omega(x)|=\frac{3}{16}$$

$$\Phi(t)=3\sin(2\pi t)$$

$$\Phi'(t)=6\pi\cos(2\pi t)$$

$$\Phi''(t)=-12\pi^2\sin 2\pi t$$

$$\Phi'''(t)=-24\pi^3\cos 2\pi t$$

$$\max_{0\leq t\leq 1}|\Phi'''(t)|=24\pi^3$$

$$\max_{0\leq x\leq 1}|p_2(x)-\phi(x)|=$$

A) $\frac{1}{6}\cdot\frac{9}{16}\cdot 24\pi^3=\frac{9}{4}\pi^3\approx 69.76$

B) $\frac{1}{6}\cdot\frac{3}{16}\cdot 24\pi^3=\frac{3}{4}\pi^3\approx 23.2547$

Costruzione della formula di quadratura di tipo interpolatorio.

1° modo

$$\tilde{I}(f) = a f(0) + b f\left(\tfrac{1}{4}\right) + c f\left(\tfrac{3}{4}\right)$$

$r=0$ $\quad \int_0^1 1\,dx = 1 \qquad a+b+c=1$

$f(x)=1$

$r=1$ $\quad \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} \qquad b\cdot\frac{1}{4} + c\cdot\frac{3}{4} = \frac{1}{2} \Rightarrow b+3c=2$

$f(x)=x$

$r=2$ $\quad \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \qquad b\cdot\frac{1}{16} + c\cdot\frac{9}{16} = \frac{1}{3} \Rightarrow 3b+27c=16$

$f(x)=x^2$

$$\begin{cases} a+b+c=1 \\ b+3c=2 \quad \times 3 \\ 3b+27c=16 \end{cases} \qquad \begin{array}{r} 3b+9c=6 \\ 3b+27c=16 \\ \hline 18c=10 \end{array} \qquad c=\frac{5}{9}$$

$$b = 2-3c = 2-\frac{15}{9} = \frac{1}{3}$$

$$a = 1-\frac{1}{3}-\frac{5}{9} = \frac{1}{9}$$

$$\tilde{I}(f) = \frac{1}{9} f(0) + \frac{1}{3} f\left(\tfrac{1}{4}\right) + \frac{5}{9} f\left(\tfrac{3}{4}\right)$$

2° modo

$$\tilde{I}(f) = \sum_{i=0}^{2} \alpha_i f(x_i)$$

$$x_0 = 0 \,, \quad x_1 = \frac{1}{4} \,, \quad x_2 = \frac{3}{4}$$

$$\alpha_i = \int_0^1 L_i(x)\,dx$$

- $$\alpha_0 = \int_0^1 L_0(x)\,dx = \int_0^1 \frac{\left(x-\frac{1}{4}\right)\left(x-\frac{3}{4}\right)}{\left(0-\frac{1}{4}\right)\left(0-\frac{3}{4}\right)}\,dx = \frac{16}{3}\int_0^1 \left(x^2 - x + \frac{3}{16}\right)dx =$$

$$= \frac{16}{3}\left|\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{16}x\right|_0^1 = \frac{16}{3}\left(\frac{1}{3} - \frac{1}{2} + \frac{3}{16}\right) = \frac{\cancel{16}}{3}\cdot\frac{16-24+9}{\cancel{48}_{\,3}} = \frac{1}{9}$$

- $$\alpha_1 = \int_0^1 L_1(x)\,dx = \int_0^1 \frac{x\left(x-\frac{3}{4}\right)}{\left(\frac{1}{4}-0\right)\left(\frac{1}{4}-\frac{3}{4}\right)}\,dx = \int_0^1 \frac{x^2 - \frac{3}{4}x}{\frac{1}{4}\left(-\frac{1}{2}\right)}\,dx =$$

$$= -8\left|\frac{x^3}{3} - \frac{3}{4}\frac{x^2}{2}\right|_0^1 = -8\left(\frac{1}{3} - \frac{3}{8}\right) = -\cancel{8}\,\frac{8-9}{\cancel{24}_{\,3}} = \frac{1}{3}$$

- $$\alpha_2 = \int_0^1 L_2(x)\,dx = \int_0^1 \frac{x\left(x-\frac{1}{4}\right)}{\left(\frac{3}{4}-0\right)\left(\frac{3}{4}-\frac{1}{4}\right)}\,dx = \int_0^1 \frac{x^2 - \frac{1}{4}x}{\frac{3}{4}\cdot\frac{1}{2}}\,dx =$$

$$= \frac{8}{3}\left|\frac{x^3}{3} - \frac{1}{4}\frac{x^2}{2}\right|_0^1 = \frac{8}{3}\left(\frac{1}{3} - \frac{1}{8}\right) = \frac{8}{3}\,\frac{5}{24} = \frac{5}{9}$$

Grado di precisione

$r=0 \qquad \frac{1}{9}+\frac{1}{3}+\frac{5}{9}=1$ OK

$r=1 \qquad \frac{1}{3}\cdot\frac{1}{4}+\frac{5}{9}\cdot\frac{3}{4}=\frac{1}{2}$ OK

$r=2 \qquad \frac{1}{3}\cdot\frac{1}{16}+\frac{5}{\cancel{9}}\cdot\frac{\cancel{9}}{16}=\frac{1}{3}$

$r=3$

$f(x)=x^3$

$$\frac{1}{9}\cdot 0^3+\frac{1}{3}\cdot\left(\frac{1}{4}\right)^3+\frac{5}{9}\left(\frac{3}{4}\right)^3=\frac{1}{3}\cdot\frac{1}{64}+\frac{5}{\cancel{9}}\cdot\frac{\overset{3}{\cancel{27}}}{64}=$$

$$=\frac{1}{192}+\frac{15}{64}=\frac{1+45}{192}=$$

$$\frac{46}{192}=\frac{23}{96}$$

$$\int_0^1 x^3\,dx=\left.\frac{x^4}{4}\right|_0^1=\frac{1}{4}$$

$\frac{1}{4}\neq\frac{23}{96} \Rightarrow$ G.P. 2

3) Stimare il numero minimo di sottointervalli di uguale ampiezza in cui si deve suddividere l'intervallo $[0, 2\pi]$, affinchè l'errore che si commette interpolando con una spline lineare la funzione $f(x) = e^{-x}\cos x$ sia minore di 10^{-2}. Trovare la formula di quadratura composita corrispondente. Di che formula si tratta?

I° itinere

17|11|2011

$[0, 2\pi]$ $\quad h = \frac{2\pi}{n}$ $\quad |f(x) - s_1(x)| \leq \frac{h^2}{12} \max_{0 \leq t \leq 2\pi} |f''(t)|$

$$f(x) = e^{-x}\cos x$$

$$f'(x) = -e^{-x}\cos x + e^{-x}(-\operatorname{sen} x) = -e^{-x}(\cos x + \operatorname{sen} x)$$

$$f''(x) = e^{-x}(\cos x + \operatorname{sen} x) - e^{-x}(-\operatorname{sen} x + \cos x) =$$

$$= e^{-x}(\cos x + \operatorname{sen} x + \operatorname{sen} x - \cos x) =$$

$$= 2e^{-x}\operatorname{sen} x$$

$\downarrow$ per trovare $\max |f''(x)|$

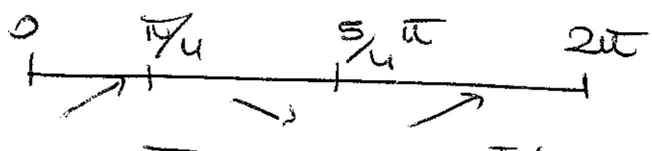
$$f'''(x) = -2e^{-x}\operatorname{sen} x + 2e^{-x}\cos x =$$

$$= -2e^{-x}(\operatorname{sen} x - \cos x) \geq 0$$

$$2e^{-x}(\operatorname{sen} x - \cos x) \leq 0$$

$x \in [0, 2\pi]$ $\quad \operatorname{sen} x \leq \cos x$

$0 \leq x \leq \frac{\pi}{4} \quad \cup \quad \frac{5}{4}\pi \leq x \leq 2\pi \qquad f''(0) = f''(2\pi) = 0$



$$|f''(\tfrac{\pi}{4})| = 2e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2} = e^{-\pi/4}\sqrt{2}$$

$$|f''(\tfrac{5}{4}\pi)| = |2e^{-\frac{5}{4}\pi}(-\frac{\sqrt{2}}{2})| = e^{-\frac{5\pi}{4}} \cdot \sqrt{2}$$

$$\max_{0 \leq x \leq 2\pi} |f''(x)| = 2e^{-\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{e^{\pi/4}}$$

Stima dell' errore:

$$\frac{1}{8}\left(\frac{2\pi}{n}\right)^2 \cdot \frac{\sqrt{2}}{e^{\pi/4}} < 10^{-2}$$

$$\frac{\pi^2}{2n^2} \cdot \frac{\sqrt{2}}{e^{\pi/4}} \cdot 100 < 1$$

$$n^2 > \frac{100\pi^2}{\sqrt{2}e^{\pi/4}} \qquad n > \frac{10\pi}{\sqrt{\sqrt{2}e^{\pi/4}}} \approx 17{,}83$$

$$\bar{n} = 18$$