# INTRODUCTION TO NETS

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## 1. Sequences do not describe the topology

The goal of this first part is to justify via some examples the fact that sequences are not sufficient to describe a topological space. In order to do so, we start discussing metric spaces in which sequences are indeed sufficient in the following sense: if two metrics $d_1$ and $d_2$ on the same set $X$ result in the same convergent sequences to the same limits, then they also induce the same topology; thus, knowing the behavior of sequences implies the knowledge of the topology. In reality it is not even necessary to ask the two limits to coincide, since it can be deduced: i.e. $x_n \overset{d_1}{\to} x$ and $x_n \overset{d_2}{\to} \widetilde{x}$ implies $x = \widetilde{x}$. This is because the sequence[1] $x_1, x, x_2, x, x_3, x, \cdots$ converges in $d_1$ to $x$, so it must converge also in $d_2$. However its subsequence $x_1, x_2, x_3, \cdots$ converges in $d_2$ to $\widetilde{x}$, while $x, x, x, \cdots$ converges in $d_2$ to $x$, hence $x = \widetilde{x}$. Now the argument can easily proceed in at least two ways:

- By our assumption the identity $id : (X, d_1) \longrightarrow (X, d_2)$ is sequentially bicontinuous, so is bicontinuous (since in metric spaces continuity equals sequential continuity), hence the two topologies are equal.
- We prove that the two topologies have the same closed sets. $S \subseteq X$ is $d_1$-closed iff it contains all $d_1$-limits of all sequences in $S$; however they are exactly all $d_2$-limits of all sequences in $S$, so $S$ is $d_2$-closed. Briefly, $d_1$-closed iff $d_1$-sequentially closed iff $d_2$-sequentially closed iff $d_2$-closed.

So we get that in metric spaces sequences are enough and the reason for this are the equivalences: continuous $\iff$ sequentially continuous and closed $\iff$ sequentially closed. Now we look at topological spaces and we see that in general nothing of this still holds.

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These notes are both the first notes and the first text in English for me, so almost surely you will find many mistakes of various sorts. I would really appreciate any suggestion of improvement or correction of errors. Thank you!

[1] this is not really a way to write sequences, but you understand what I mean.

**Example 1.1.** $(\ell_1, w)$.

It's well known that this is a Shur space, i.e. a sequence $\{x_n\}_{n=1}^{\infty}$ in $\ell_1$ converges weakly iff it converges in norm. So we see that two different topologies (the one induced by the norm and the weak one) result in the same convergent sequences, in particular $w$-convergence is not sufficient to describe the $w$-topology. Further

$$id : (\ell_1, w) \longrightarrow (\ell_1, \|.\|)$$

is sequentially continuous but not continuous, since the weak topology is strictly weaker than the norm one.

We also have differences for closed sets: $S_{\ell_1}$ is strongly sequentially closed, so weakly sequentially closed, but $\overline{S_{\ell_1}}^{w} = B_{\ell_1}$ (this equality holds in every infinite dimensional Banach space). Thus we have

$$\overline{S_{\ell_1}}^{seq-w} = S_{\ell_1} \subsetneqq B_{\ell_1} = \overline{S_{\ell_1}}^{w}.$$

Our next example is even more elementary in that requires no Functional Analysis; however is somewhat artificial.

**Example 1.2.** $([0,1], \tau)$ [Megginson] Ex. 2.3.

Consider the following topology on the unit interval

$$\tau := \{A \subseteq [0,1] : 0 \notin A \text{ or } \mathrm{card}\,([0,1] \setminus A) \leq \aleph_0\}.$$

Clearly this is an Hausdorff topology and $\{0\} \notin \tau$, so that $(0,1]$ is not closed. However $(0,1]$ is sequentially closed, since no sequence in $(0,1]$ can converge to 0. Indeed let $\{x_n\}_{n=1}^{+\infty} \subseteq (0,1]$ and note that $U := [0,1] \setminus \{x_n\}_{n=1}^{+\infty} \in \tau$ is a neighborhood of 0 disjoint from the given sequence.

*Remark* 1.1. Maybe it can be interesting to note that the reason why this example works is that there are too many neighborhoods of 0 and they are all too big: they are so many that can elude a given sequence, but all are too big to elude the whole $(0,1]$. This is mathematically expressed by the fact that 0 does not admit a countable local basis, as it is immediate to see.

Actually one can show that missing a countable local basis is just what can go wrong, since in first countable spaces[2] continuity equals sequential continuity and closedness sequential closedness (proofs are very similar to those for metric spaces). Hence sequences are sufficient not only in metric spaces, but also in first countable ones.

This generalization is nevertheless not much interesting in Functional Analysis since one can show that for linear topologies metrizability is *equivalent* to first countability (see [Rudin] 1.24), so for linear topologies this is actually not a more general case.

[2] i.e. every point admits a countable local basis.

These two examples show that the equivalences previously stated fail in topological spaces; still, one can be not much satisfied with the first one since Shur spaces are quite pathological examples and one can suspect that in "honest" spaces sequences should do their work. This is tragically false:

**Exercise 1.1.** [Rudin] Chp. 3 Ex. 9 (this example is due to Von Neumann).

Consider $\ell_2$ with the usual complete orthonormal system $\{e_n\}_{n=0}^{+\infty}$, let $f_{a,b} := e_a + ae_b$, $a, b \in \mathbb{N}$ and $E := \{f_{a,b} : 0 \leq a < b\}$.

(1) find all elements of $\overline{E}^{seq-w}$;
(2) find all elements of $\overline{E}^{w}$;
(3) show that $0 \in \overline{E}^{w}$, $0 \notin \overline{E}^{seq-w}$, but $0 \in \overline{\left(\overline{E}^{seq-w}\right)}^{seq-w}$.

This is a very main example since it shows that even in the most easy to handle space (separable Hilbert space) sequences are not sufficient and even more:

$$\overline{E}^{seq-w} \subsetneq \overline{\left(\overline{E}^{seq-w}\right)}^{seq-w}.$$

This says that the sequential closure of a set might be not sequentially closed (probably it was already clear, but recall that the sequential closure of $E$ is the set of all limits of sequences in $E$). This implies that

$$E \longmapsto \overline{E}^{seq-w}$$

is *not* a closure operator, i.e. it doesn't correspond to the closure in any topology. More explicitly, there exists in general no topology whose closed sets are exactly the sequentially closed ones!

This shows once for all that sequences are not powerful enough and motivates the need to look for some more general concept.

## 2. Nets

Here we introduce this generalization of the concept of sequences and we prove the most basic properties.

There is a very simple idea which is important to underline: we want points of the closure of a set to be limit of these nets. In order to get this we want to generalize the proof given for metric spaces, with balls of radius $1/n$, and the one for first countable spaces, using a countable family of neighborhoods; in these, to any of this neighborhoods one associates a point in the set in such a way that the resulting sequence converges, so at the end of the story what one does is using the countable local basis as index for the sequence in a clever way so that when the neighborhoods shrink to the point the sequence converges to the point itself.

With sequences this is latent, while nets are exactly the mathematical way to make this straight, as will be apparent from the first proof on.

**Definition 2.1.** Let $I \neq \emptyset$ be a set. A *preorder* on $I$ is a relation $\leq$ that satisfies:

(1) reflexivity: $\alpha \leq \alpha$ for each $\alpha \in I$;
(2) transitivity: $\alpha \leq \beta$, $\beta \leq \gamma \Rightarrow \alpha \leq \gamma$ for each $\alpha, \beta, \gamma \in I$.

**Definition 2.2.** A *directed set* is a nonempty set $I$ with a preorder $\leq$ such that for every $\alpha, \beta \in I$ these exists $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

**Definition 2.3.** A *net* in $X$ ($X$ is just a set here) is a mapping from $I$ to $X$. Usually it is denoted

$$\begin{array}{rcl} I & \longrightarrow & X \\ \alpha & \longmapsto & x(\alpha) \equiv x_\alpha \end{array}$$

or briefly $(x_\alpha)_{\alpha \in I}$ or even $(x_\alpha)$ without mention to the directed set when it causes no confusion.

**Example 2.1.** $\mathbb{N}$ and $\mathbb{R}$ are clearly directed sets with the usual orders. Then any $\varphi : \mathbb{N} \longrightarrow X$ is a net, so every sequence is a net. Also, every $\varphi : \mathbb{R} \longrightarrow X$ is a net. In particular nets may have no first element and the index family may be more than countable, but may also be finite.

We now need convergence of nets and in order to do so we need a topology on $X$. So from now with $X$ we will denote a topological space $(X, \tau)$ and $\mathcal{U}_x$ will denote the family of all neighborhoods of $x$. Let us also recall the definition of subbasis for a topology: a family $\mathscr{S}$ of subsets of $X$ is a *subbasis* for $\tau$ if the collection of all finite intersections of elements in $\mathscr{S}$ produces a basis for the topology.

**Definition 2.4.** Let $(x_\alpha)$ be a net in $X$ and let $x \in X$. Then we say that $(x_\alpha)$ *converges* to $x$, written $x_\alpha \overset{\alpha}{\to} x$ if for every $U \in \mathcal{U}_x$ there is $\alpha_0 \in I$ such that for all $\alpha \geq \alpha_0$ one has $x_\alpha \in U$. Obviously here $\alpha \geq \alpha_0$ means by definition $\alpha_0 \leq \alpha$.

*Remark* 2.1. If $(x_\alpha)$ is a sequence, as in Example 2.1, then this definition is just the definition of convergence of sequences.

Now we turn to some basic result; the first one in particular says that we do not really need to check the previous condition on *all* neighborhoods.

**Proposition 2.1.** (subbasis and convergence).

(1) $x_\alpha \overset{\alpha}{\to} x$ *iff the condition of the definition holds for each* $U$ *in a subbasis of neighborhoods of* $x$.

(2) *A net in a topological product converges iff all its coordinates converge; i.e. the product topology is the topology of coordinatewise convergence.*

*Proof.* (1) $\Longrightarrow$ is clear since we have to test the condition on less neighborhoods.

$\Longleftarrow$ Let $U \in \mathcal{U}_x$, so that there exist $U_1, \cdots, U_N \in \mathcal{U}_x$ elements of the given subbasis with $U_1 \cap \cdots \cap U_N \subseteq U$ ($U$ contains an element of the basis and this is finite intersection of elements from the subbasis). On each $U_i$ we use the condition of Definition 2.4: for $i = 1, \cdots, N$ there exists $\alpha_0^i \in I$ such that for all $\alpha \geq \alpha_0^i$ one has $x_\alpha \in U_i$. Now we use the definition of a directed set and induction: there exists $\alpha_0 \in I$ with $\alpha_0^i \leq \alpha_0$ for all $i = 1, \cdots, N$, so by transitivity for every $\alpha \geq \alpha_0$ one has $x_\alpha \in \cap_{i=1}^N U_i \subseteq U$. This proves that for $U$ the condition of Definition 2.4 holds.

(2) Let $X^{(\beta)}$ be topological spaces and let $X := \prod_{\beta \in B} X^{(\beta)}$ endowed with the product topology[3]; recall that the product topology is defined giving a subbasis, which is the following:

$$\mathscr{S} := \left\{ \prod_{\beta \in B} U^{(\beta)} : U^{(\beta)} \text{ is open in } X^{(\beta)} \text{ and at most one is strictly contained in } X^{(\beta)} \right\}$$

or, which is the same

$$\mathscr{S} := \left\{ U^{(\beta_0)} \times \prod_{\substack{\beta \in B \\ \beta \neq \beta_0}} X^{(\beta)} : \beta_0 \in B \text{ and } U^{(\beta_0)} \text{ is open in } X^{(\beta_0)} \right\}.$$

But clearly, given a net $(x_\alpha) \in X$ we have that

$$x_\alpha \in U^{(\beta_0)} \times \prod_{\substack{\beta \in B \\ \beta \neq \beta_0}} X^{(\beta)}$$

iff $x_\alpha^{(\beta_0)} \in U^{(\beta_0)}$; thus testing convergence on all elements of $\mathscr{S}$ is exactly testing coordinatewise convergence. However $\mathscr{S}$ is a subbasis, so by 1. the convergence of $(x_\alpha)$ in $X$ is coordinatewise convergence. $\square$

**Theorem 2.1.** (nets describe the topology).

[3]here the indexes of the topological product are superscripts, while indexes of nets are subscripts. There is no deep reason for that, it's just to distinguish.

(1) *$X$ is an Hausdorff space $\iff$ every net admits at most one limit;*
(2) *$f : X \longrightarrow Y$ is continuous $\iff$ is net-continuous;*
(3) *$S \subseteq X$ is closed $\iff$ is net-closed.*

We didn't give the definitions of net-continuity and of net-closedness, but they are exactly what one expects. $f : X \longrightarrow Y$ is net-continuous if it sends convergent nets into convergent nets $(x_\alpha \overset{\alpha}{\to} x \implies f(x_\alpha) \overset{\alpha}{\to} f(x))$ , $S$ is net-closed iff it contains all limits of nets in $S$.

In the statement of the theorem it's quite clear that the three $\implies$ implications are not unexpected: it's part of our intuition that continuous functions should preserve convergence. So the three implications $\implies$ of the theorem show us that this generalization of sequences is reasonable. But the three $\impliedby$ implications are much more unexpected and much more interesting: indeed we can use just the same argument we gave at beginning using now (2) or (3) to deduce that two topologies that induce the same convergent nets to the same limits must be the same. This is the first very main advantage of nets with respect to sequences, so right-to-left implications say that nets are a very interesting generalization.

*Proof.* (1) $\implies$ If by contradiction $(x_\alpha)$ is a net such that $x_\alpha \to x$ and $x_\alpha \to y$, with $x \neq y$, then there exist $U \in \mathcal{U}_x, V \in \mathcal{U}_y$ disjoint, so by definition of convergence there are $\alpha_0, \beta_0 \in I$ such that for each $\alpha \in I$ with $\alpha \geq \alpha_0$ we have $x_\alpha \in U$ and for each $\alpha \in I$ with $\alpha \geq \beta_0$ we have $x_\alpha \in V$. Let $\gamma \in I$ be such that $\alpha_0 \leq \gamma, \beta_0 \leq \gamma$, so that for $\alpha \geq \gamma$ we have $x_\alpha \in U \cap V = \emptyset$; this is a contradiction.

$\impliedby$ Assume $X$ is not Hausdorff, so there are $x \neq y$ in $X$ such that all neighborhoods of $x$ intersect all neighborhoods of $y$. We want to prove that there exists a net with at least two limits. We build it in this way. Consider as a set $\mathcal{U}_x \times \mathcal{U}_y$ and the following order relation on it:

$$(U_x, U_y) \preceq (V_x, V_y) \iff [U_x \supseteq V_x \wedge U_y \supseteq V_y]$$

(note that the inclusion is *reversed*). Since intersection of two neighborhoods is a neighborhood, this is a directed set. But for each $(U_x, U_y) \in \mathcal{U}_x \times \mathcal{U}_y$ we have $U_x \cap U_y \neq \emptyset$, so we can choose (here we need to use the Axiom of Choice) $x_{(U_x,U_y)} \in U_x \cap U_y$ for each $(U_x, U_y) \in \mathcal{U}_x \times \mathcal{U}_y$. In this way we have found a net:

$$\left(x_{(U_x,U_y)}\right)_{(U_x,U_y)\in\mathcal{U}_x\times\mathcal{U}_y}.$$

Now we just need to check that this net converges to both $x$ and $y$. We prove it for $x$ since for $y$ is obviously the same argument. Fixed $U \in \mathcal{U}_x$ we use as the $\alpha_0$ of the definition of convergence the element $(U, X) \in \mathcal{U}_x \times \mathcal{U}_y$ and it is obvious that for each $(U_x, U_y) \in \mathcal{U}_x \times \mathcal{U}_y$ with $(U_x, U_y) \succeq (U, X)$ we have

$x_{(U_x,U_y)} \in U_x \cap U_y \subseteq U \cap X = U$; thus we have that for every neighborhood of $x$ there exists an index such that the net lies in that neighborhood for all indexes bigger than the given one, which is the definition of convergence.

(2) $\Longrightarrow$ Let $x_\alpha \xrightarrow{\alpha} x$, we want to prove that $f(x_\alpha) \xrightarrow{\alpha} f(x)$. Since $f$ is continuous, for any $U \in \mathcal{U}_{f(x)}$, we have that $f^{-1}(U) \in \mathcal{U}_x$; hence there is $\alpha_0 \in I$ such that for each $\alpha \geq \alpha_0$ we have $x_\alpha \in f^{-1}(U)$, so that $f(x_\alpha) \in U$.

$\Longleftarrow$ If $f$ is not continuous, then there exists $U$ open in $Y$ such that $f^{-1}(U)$ is not open in $X$, so there exists $x_0 \in f^{-1}(U)$ which is not an interior point, while $f(x_0) \in U$ is interior. By definition this means that every neighborhood of $x_0$ intersects $\left(f^{-1}(U)\right)^{\complement}$, so given $V \in \mathcal{U}_{x_0}$ there is $x_V \in V \cap \left(f^{-1}(U)\right)^{\complement}$. As before, $\mathcal{U}_{x_0}$ with the reversed inclusion is a directed set, so $(x_V)$ is a net which converges to $x_0$ (exactly the same argument of (1)); however $f(x_V) \notin U$ so that $f(x_V) \stackrel{\alpha}{\nrightarrow} f(x)$ and $f$ is not net-continuous.

(3) $\Longrightarrow$ Let $x_0 \in X$ and $(x_\alpha) \subseteq S$ be such that $x_\alpha \xrightarrow{\alpha} x_0$, we want to prove that $x_0 \in S$. But every neighborhood of $x_0$ contains elements of $(x_\alpha)$, so intersects $S$, hence for each $U \in \mathcal{U}_{x_0}$ one has $U \cap S \neq \emptyset$. This means exactly that $x_0 \in \overline{S} = S$ since $S$ is closed.

$\Longleftarrow$ Let $x_0 \in \overline{S}$, we show that $x_0 \in S$ by proving that there exists a net in $S$ which converges to $x_0$. This is now a very familiar argument: for each $U \in \mathcal{U}_{x_0}$ there is $x_U \in U \cap S$ and, looking at $\mathcal{U}_{x_0}$ with the reverse inclusion, this $(x_U)$ is a net in $S$ that converges to $x_0$. $\square$

*Remark* 2.2. The proof of all left-to-right implications has really nothing new, the right-to-left implications are based on a very important idea: neighborhoods with reversed inclusion are a directed set and after this remark the proof becomes just an easy verification. This idea is exactly the one we underlined at the beginning of the section and after this proof it should be clear why nets are a very natural generalization of sequences.

## 3. Subnets and Compactness

It would have been very nice to find in Theorem 2.1 a statement giving a characterization of compactness in terms of nets, for example something like "$X$ is compact iff is net-compact". However it's clear that there is at least one step missing. Indeed "net-compactness" should generalize sequential compactness, so it should require some concept of a "sub-net" of a net, which we did not introduce. We now try a first definition, which may seem quite natural but is in reality far from what we need. In analogy with sequences one could try to define a subnet of a net $(x_\alpha)$ as the restriction of the net to a suitable subset $J$ of $I$, which should contain elements of $I$ with "large" indexes. For example we can say:

**Definition 3.1.** A subset $J$ of a directed set $I$ is said to be *cofinal* if for each $\alpha \in I$ there is $\beta \in J$ with $\alpha \leq \beta$.

Then one could define a subnet of the net $(x_\alpha)$ to be the restriction of the net to a cofinal subset of $I$. Note that with this definition any subnet of a sequence is indeed a subsequence of the given sequence and this could seem to be a very good aspect of our definition. In reality this is a problem: if we want compactness to be equivalent to net-compactness, then in particular every *sequence* in a compact topological space must admit a convergent subnet, so a convergent subsequence. However it's known that this is false (see *Remark* 3.5); thus we have to accept that sequences may admit subnets that are not subsequences.

In reality the definition of subsequence itself is not to restrict the sequence to a subset, but is a pre-composition with a strictly increasing $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$. Here we do not have a concept of strictly increasing functions (we only have preorders), but note that if $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ is strictly increasing, then it is increasing and $\varphi(\mathbb{N})$ is cofinal in $\mathbb{N}$; these two properties can be used for a definition in every directed set. With this the definition of subnet is almost ready, we just need to avoid the mistake of using the same directed set (for the same reason of before) and we get:

**Definition 3.2.** Let $\varphi : I \longrightarrow X$ be a net and let $J$ be a directed set. If $\psi : J \longrightarrow I$ is a mapping such that:

(1) $\beta_1 \leq \beta_2$ in $J$ implies $\psi(\beta_1) \leq \psi(\beta_2)$ in $I$,
(2) $\psi(J)$ is cofinal in $I$,

then we say that $\varphi \circ \psi : J \longrightarrow X$ is a *subnet of* $\varphi$.

If the net $\varphi$ is denoted $(x_\alpha)_{\alpha \in I} \equiv (x_\alpha)$ then $\varphi \circ \psi$ is denoted $(x_{\psi(\beta)})_{\beta \in J} \equiv (x_{\psi(\beta)})$, sometimes even $(x_\beta)$ if no confusion can arise.

First of all we need an example to convince us that with this definition sequences may admit subnets that are not subsequences.

**Example 3.1.** Let $(x_n)$ be a sequence and let $\psi : [1, +\infty) \longrightarrow \mathbb{N}$ defined by $r \longmapsto \psi(r) := \lfloor r \rfloor$. It's clear that this $\psi$ satisfies the two conditions (for (2) simply note that $\psi$ is onto), so $(x_{\psi(r)})_{r \in [1,+\infty)}$ is a subnet of $(x_n)$. Obviously this is not a sequence, in particular passing to a subnet may increase the set of indexes.

*Remark* 3.1. If a sequence converges to a limit, then all its subsequences converge to the same limit. Luckily, the same holds for nets. Indeed if $x_\alpha \to x$ and $(x_{\psi(\beta)})$ is a subnet, then for every $U \in \mathcal{U}_x$ there exists $\alpha_0 \in I$ such that for each $\alpha \geq \alpha_0$ one has $x_\alpha \in U$. By cofinality, there exists $\beta_0 \in J$

such that $\psi(\beta_0) \geq \alpha_0$, so for $\beta \geq \beta_0$ one has $\psi(\beta) \geq \psi(\beta_0) \geq \alpha_0$ which gives $x_{\psi(\beta)} \in U$.

*Remark* 3.2. One could give a slightly more general definition of subnet, for example see [Kelley] where $\psi$ is required to satisfy, instead of (1) and (2), this condition: for each $\alpha_0 \in I$ there is $\beta_0 \in J$ such that for every $\beta \geq \beta_0$ one has $\psi(\beta) \geq \alpha_0$.

For what we need here, there is no real difference, just some small modifications in proofs.

For the proof of the equivalence between compactness and net-compactness it is useful to introduce the concept of *accumulation point*, which as usual generalizes the corresponding definition for sequences.

**Definition 3.3.** A net $(x_\alpha)$ *accumulates at a point* $x_0$ or $x_0$ is an *accumulation point for* $(x_\alpha)$ if for every $U \in \mathcal{U}_{x_0}$ and every $\alpha_0 \in I$ there is $\alpha \in I$ with $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.

*Remark* 3.3. It's clear that if a net converges to a point, then it accumulates at that point. Moreover we have that if a subnet accumulates at a point, then all the net accumulates at the point. Let us prove this second statement: assume $(x_{\psi(\beta)})$ accumulates at $x$ and fix $U \in \mathcal{U}_x$ and $\alpha_0 \in I$. By cofinality there is $\beta_0 \in J$ with $\psi(\beta_0) \geq \alpha_0$ and since $(x_{\psi(\beta)})$ accumulates at $x$ we also have that there exists $\beta \in J, \beta \geq \beta_0$ with $x_{\psi(\beta)} \in U$. By monotonicity $\alpha := \psi(\beta) \geq \alpha_0$ is the index we need.

Putting together these two facts we get that if a subnet converges to a point, then the net accumulates at the point and the first serious result on accumulation points is that the converse holds here. But before proving it, let's note that surprisingly this is false with sequences![4]

**Example 3.2.** $(\ell_1, w)$ again.

Consider a sequence dense in the unit sphere $S_{\ell_1}$ with respect to the norm topology. Of course no subsequence can converge weakly to 0 (it is Shur), but this sequence accumulates at 0. Indeed any weak neighborhood $U$ of 0 intersects the unit sphere and $U \cap S_{\ell_1}$ is open in the restriction of the norm topology to the sphere. By density the sequence must intersect this open set infinitely many times.

Note that also Exercise 1.1 provides a counterexample to this.

**Theorem 3.1.** (subnets and accumulation).

*A net accumulates at a point iff some subnet converges to that point.*

[4]Once again, this equivalence is true in metric spaces but not in topological ones.

*Proof.* $\Longleftarrow$ Is part of *Remark* 3.3.

$\Longrightarrow$ Is very similar to the previous proofs, we just need a slightly more careful choice of the directed set. Let $(x_\alpha)$ be a net that accumulates at $x_0$ and let

$$J := \{(\alpha, U) \in I \times \mathcal{U}_{x_0} : x_\alpha \in U\}$$

with the preorder given by $(\alpha_1, U_1) \preceq (\alpha_2, U_2) \iff \alpha_1 \leq \alpha_2$ and $U_1 \supseteq U_2$. Since $(x_\alpha)$ accumulates at $x_0$, this is a directed set: indeed given $(\alpha_1, U_1), (\alpha_2, U_2)$ there exists $\gamma \in I$ with $\alpha_1 \leq \gamma$, $\alpha_2 \leq \gamma$, so there exists $\alpha_3 \in I$, $\alpha_3 \geq \gamma$ such that $x_{\alpha_3} \in U_1 \cap U_2$. Hence $(\alpha_3, U_1 \cap U_2) \in J$ and $(\alpha_i, U_i) \preceq (\alpha_3, U_1 \cap U_2)$, $i = 1, 2$. Then the mapping $\psi : J \longrightarrow I$ given by $\psi(\alpha, U) := \alpha$ defines a subnet $(x_{\psi(\alpha,U)})$ which converges to $x_0$ (indeed for every $U_0 \in \mathcal{U}_{x_0}$ there exists $\alpha_0 \in I$ such that $x_{\alpha_0} \in U_0$, so for $(\alpha, U) \succeq (\alpha_0, U_0)$ we have $x_{\psi(\alpha,U)} \in U_0$). $\square$

We are finally ready for the result we were looking for.

**Theorem 3.2.** (compactness).

*$X$ is compact iff every net in $X$ admits a convergent subnet (i.e. $X$ is net-compact) iff every net in $X$ accumulates at some point of $X$.*

Since compactness is an intrinsic property, this immediately implies that $S \subseteq X$ is compact iff every net in $S$ admits a subnet converging in $S$ (i.e. $S$ is net-compact) iff every net in $S$ accumulates at some point of $S$.

*Proof.* The equivalence between the second and the third statement follows obviously from Theorem 3.1. We now prove the equivalence between the first and the third.

$\Longrightarrow$ By contradiction, let $(x_\alpha)$ be a net in $X$ with no accumulation point in $X$. Then fixed $x \in X$, $(x_\alpha)$ doesn't accumulate on $x$, so there are $U_x \in \mathcal{U}_x$ and $\alpha_x \in I$ such that $x_\alpha \notin U_x$ for all $\alpha \geq \alpha_x$.

Now $\{U_x\}_{x \in X}$ is an open cover of a compact set, so there are finitely many points $x_1, \cdots, x_N$ such that $X = \cup_{i=1}^N U_{x_i}$. Let $\beta \geq \alpha_{x_i}$, $i = 1, \cdots, N$ so that $x_\beta \notin U_{x_i}$, $i = 1, \cdots, N$ and then $x_\beta \notin \cup_{i=1}^N U_{x_i} = X$.

$\Longleftarrow$ By contraposition: if $X$ is not compact, then there exists an open cover $\mathcal{O}$ with no finite subcover. Then the family

$$\mathscr{G} := \{G \subseteq X : G \text{ is finite union of elements of } \mathcal{O}\}$$

is a directed set with the inclusion (*not* reversed!) and by assumption each element of $\mathscr{G}$ is strictly contained in $X$; hence we can choose[5] $x_G \in X \setminus G$. Then $(x_G)_{G \in \mathscr{G}}$ is a net and it has no accumulation point in $X$. Indeed if

[5]here and in all previous arguments the Axiom of Choice is needed.

$x \in X$ there exists $G \in \mathscr{G}$ such that $x \in G$, so for every $H \geq G$ we have that $x_H \notin H$; thus $x_H \notin G$ gives that $(x_G)_{G\in\mathscr{G}}$ does not accumulate at $x$. □

*Remark* 3.4. In all these proofs the point is always the same: we use neighborhoods as a directed set, from the assumptions we deduce information about some neighborhoods, we translate them looking neighborhoods as indexes and this gives properties of the net. After this remark all proofs reduce just to the verification of some small detail.

*Remark* 3.5. Theorem 3.2 is a very important characterization of compactness which generalizes the well known result for metric spaces. Its importance is also due to the fact that the sequential characterization fails to hold in topological spaces, since both implications are false. For example, by the Banach-Alaoglu Theorem $B_{\ell_\infty^*}$ is $w^*$-compact , but is easy to see that it is not $w^*$-sequentially compact. For a counterexample to the other implication consider the following very beautiful example.

**Example 3.3.** [Manetti] Exercise 7.14.

Consider the topological product $[0,1]^{\mathbb{R}}$, i.e. the set of all functions $f : \mathbb{R} \longrightarrow [0,1]$ with the pointwise convergence topology. Let

$$B := \left\{ f \in [0,1]^{\mathbb{R}} : \operatorname{card}\{f \neq 0\} \leq \aleph_0 \right\}.$$

Then we have that:

(1) $B$ is dense in $[0,1]^{\mathbb{R}}$;
(2) $B$ is not compact;
(3) $B$ is sequentially compact.

Indeed for 1, let $f \in [0,1]^{\mathbb{R}}$ and let $U \in \mathcal{U}_f$. Assume without loss of generality that $U$ is an element of the basis for the product topology, so that

$$U = U(x_1, \cdots, x_n, \epsilon; f) := \left\{ g \in [0,1]^{\mathbb{R}} : |g(x_i) - f(x_i)| < \epsilon,\ \forall i = 1, \cdots, n \right\}.$$

We want to show that this neighborhood intersects $B$ and this is obvious: indeed consider $\widetilde{f} \in [0,1]^{\mathbb{R}}$ defined by

$$\widetilde{f}(x) = \begin{cases} f(x) & x \in \{x_i\}_{i=1}^n \\ 0 & \text{otherwise} \end{cases}.$$

Obviously $\widetilde{f} \in B$ and moreover $\widetilde{f} \in U$ since $\left|\widetilde{f}(x_i) - f(x_i)\right| = |f(x_i) - f(x_i)| = 0 < \epsilon$, so that $B$ is dense. Since of course $B \subsetneqq [0,1]^{\mathbb{R}}$, $B$ is not closed, hence not compact (here we use that the product topology in $[0,1]^{\mathbb{R}}$ is Hausdorff).

Let us now prove that $B$ is sequentially compact. If $\{f_n\}$ is a sequence in $B$, we denote $F_n := \{f_n \neq 0\}$ and $F := \cup_{n=1}^{+\infty} F_n$ which is at most countable.

Hence we have that for each $x \notin F$ $\{f_n(x)\}$ is of course convergent since all terms are 0. Since $F$ is countable a straight application of the Cantor Diagonal principle produces a subsequence $\{f_{n_k}\}$ that converges in every point of $F$. So the subsequence $\{f_{n_k}\}$ converges pointwise to a certain $f$ and of course $\{f \neq 0\} \subseteq F$ so that $f \in B$.

We conclude this section with some applications of the previous results to some elementary facts; clearly they can be proved also with no use of nets.

**Example 3.4.** In a topological vector space the closure of a convex set is a convex set.

If the space is actually a normed space this is trivial, since $\overline{x}, \overline{y} \in \overline{C}$ with $C$ convex implies that there are $x_n, y_n \subseteq C$ such that $x_n \to x$, $y_n \to y$, so $\lambda x_n + (1-\lambda)y_n \in C$ converges to $\lambda\overline{x} + (1-\lambda)\overline{y}$ which therefore lies in $\overline{C}$.

But now one tries to do the same with nets and almost everything works, there is just a small detail: $\overline{x}, \overline{y} \in \overline{C}$ implies that $\exists (x_\alpha)_{\alpha\in I}$, $x_\alpha \xrightarrow{\alpha} \overline{x}$ and $\exists (y_\beta)_{\beta\in J}$, $y_\beta \xrightarrow{\beta} \overline{y}$ but in general $I \neq J$!

However one could build a couple of nets with the same directed set as domain and that converge to the same points. Indeed consider: $I \times J$ with the preorder $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ iff $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ [6], the two mappings $\psi_1 : I \times J \longrightarrow I$, $\psi_1(\alpha, \beta) := \alpha$ and $\psi_2 : I \times J \longrightarrow J$, $\psi_2(\alpha, \beta) := \beta$ and the couple of nets $(x_{\psi_1(\alpha,\beta)})_{I\times J}$ and $(x_{\psi_2(\alpha,\beta)})_{I\times J}$ are the ones we need. Then everything in the previous argument can be done the same way.

**Example 3.5.** The topological product of two compact sets is compact.

The standard proof of this fact is not completely trivial, there is some work to do, while with nets this is immediate: if $(x_\alpha, y_\alpha)_{\alpha\in I}$ is a net in $X \times Y$, then $(x_\alpha)$ is a net in the compact $X$, so there exists a convergent subnet: there is $\psi : J \longrightarrow I$ with $x_{\psi(\beta)} \xrightarrow{\beta} \overline{x}$. But now $(y_{\psi(\beta)})_{\beta\in J}$ is a net in $Y$, so there is $\xi : H \longrightarrow J$ with $y_{\psi(\xi(\gamma))} \xrightarrow{\gamma} \overline{y}$. Then the subnet $(x_{\psi(\xi(\gamma))}, y_{\psi(\xi(\gamma))})_{\gamma\in H}$ converges to $(\overline{x}, \overline{y})$.

## 4. Ultranets

Of course Example 3.5 makes us think at Tychonoff Theorem about arbitrary topological product of compact topological spaces and makes us wonder whether one could find an easy proof of the theorem by using net techniques. However it's well known that Tychonoff Theorem is equivalent to the Axiom of Choice (this was proved by Kelley), hence we can not hope to find too easy proofs, since any proof must use somewhere the Axiom of Choice or

[6] Here maybe it would be better to write $\alpha_1 \leq_I \alpha_2$ and $\beta_1 \leq_J \beta_2$.

something equivalent. Actually nets allow us to prove this result by means of the so called *ultranets* which are particularly useful in arguments that involve compactness and the Axiom of Choice. The aim of this section is to introduce them and use them to prove Tychonoff Theorem.

We begin with two definitions which we could have used from the beginning in order to define convergence and accumulation; we didn't do so since their use would have made no difference up to now, while from now on they will be quite useful.

**Definition 4.1.** A net $(x_\alpha)$ *frequents the set* $S$ $(\subseteq X)$ or *is in* $S$ *frequently* if

$$\forall \alpha \in I,\ \exists\, \beta \in I \text{ with } \alpha \leq \beta \text{ such that } x_\beta \in S.$$

**Definition 4.2.** A net $(x_\alpha)$ *is in* $S$ *ultimately* (one could use also *eventually*) if

$$\exists\, \alpha \in I,\ \forall \beta \in I \text{ with } \alpha \leq \beta \text{ such that } x_\beta \in S.$$

With these concepts it's clear that the definitions of convergence and accumulation can be rephrased as follows:

- a net converges to a point iff is ultimately in all his neighborhoods;
- a net accumulates at a point iff is frequently in all his neighborhoods.

Moreover it's clear form the definitions that:

- $(x_\alpha)$ is in $S$ ultimately iff is not in $S^{\complement}$ frequently;
- if a net is ultimately in a set, then it is frequently in that set.

Now the definition of ultranet is very natural since there is just to ask that the converse of the last statement holds.

**Definition 4.3.** An *ultranet* is a net that is ultimately in each set that frequents, i.e. a net for which the following implication holds:

$$(x_\alpha) \text{ frequents } S \Longrightarrow (x_\alpha) \text{ is in } S \text{ ultimately}.$$

If in particular $S$ is chosen to be a neighborhood of a point $x$, we have that if an ultranet is frequently in a neighborhood, then it is ultimately in that neighborhood. Hence we get:

**Lemma 4.1.** *If an ultranet accumulates at a point, then it converges to that point.*

Let now $(x_\alpha)$ be an ultranet and let $S \subseteq X$. If $(x_\alpha)$ is not ultimately in $S$, then it must be frequently in $S^{\complement}$ and being an ultranet, this implies that $(x_\alpha)$ is ultimately in $S^{\complement}$. Hence if $(x_\alpha)$ is an ultranet and $S \subseteq X$, then $(x_\alpha)$ is ultimately in $S$ or in $S^{\complement}$. Actually also the converse holds: indeed if $(x_\alpha)$ satisfies this condition and it is frequently is a set $S$, then is not ultimately in $S^{\complement}$ and by assumption this implies that is ultimately in $S$; hence $(x_\alpha)$ is an ultranet.

We have thus proved the following quite useful characterization of ultranets, which is sometimes used as a definition.

**Proposition 4.1.** *A net $(x_\alpha)$ is an ultranet iff the following holds: for every $S \subseteq X$ the net is ultimately in $S$ or in $S^{\complement}$. This is clearly also equivalent to: for every $S \subseteq X$ the net doesn't frequent both $S$ and $S^{\complement}$.*

Last basic fact we need is that ultranets are preserved by functions (no continuity assumption!).

*Remark* 4.1. If $(x_\alpha)$ is an ultranet in $X$ and $f : X \longrightarrow Y$, then $(fx_\alpha)$ is an ultranet in $Y$.

Indeed if $(fx_\alpha)$ frequents $S \subseteq Y$ , then $(x_\alpha)$ frequents $f^{-1}(S)$. By assumption this gives that $(x_\alpha)$ is ultimately in $f^{-1}(S)$, so $(fx_\alpha)$ is ultimately in $S$.

*Remark* 4.2. All the concepts introduced so far have been presented as generalizations of sequential concepts, but the one of ultranet. So is quite natural to ask whether there exists some sequential analogue of this concept. Let's try to understand which sequences are ultranets. If the range of the sequence is infinite, then we can split it into two disjoint countable sets and the sequence would frequent them both, so is not an ultranet. If the range is finite and at least two values are attained infinitely often, then these two values are frequented and again the sequence is not an ultranet. So only eventually constant sequences can be ultranets and clearly they are. Hence a sequence is an ultranet iff is eventually constant.

This remark can be interesting since it provides us the first example of an ultranet; however this is quite a trivial example and the fact that ultranet is a trivial concept for sequences makes us understand that this concept is actually interesting just for nets.

In reality we still need to show that this concept is interesting for nets and this is what we now turn to do, by showing that there are enough ultranets: every net admits a subnet which is an ultranet. Unluckily the proof of this fact is highly non constructive, since the directed set of the subnet is obtained by making use of Zorn Lemma. This is done in the following Lemma.

**Lemma 4.2.** *Let $(x_\alpha)$ be a net in $X$. Then there exists $\mathfrak{M} \subseteq 2^X$, i.e. a family of subsets of $X$ such that:*

1. *$(x_\alpha)$ frequents every element of $\mathfrak{M}$;*
2. *$\mathfrak{M}$ is closed under intersection of two elements: $M_1, M_2 \in \mathfrak{M} \Longrightarrow M_1 \cap M_2 \in \mathfrak{M}$;*

(3) *for each $S \subseteq X$ either $S$ or $S^{\complement}$ are in $\mathfrak{M}$.*

*Proof.* We begin finding a good candidate.

Consider the collection of all families of subsets that satisfy (1) and (2), i.e.

$$\mathscr{F} := \left\{ \mathfrak{N} \subseteq 2^X : \mathfrak{N} \text{ satisfies (1) and (2)} \right\}$$

and note that it is non empty since $\{X\} \in \mathscr{F}$. Partially order $\mathscr{F}$ by inclusion and note that every chain admits an upper bound (simply take the union and show that is in $\mathscr{F}$), so Zorn Lemma gives us a maximal element $\mathfrak{M} \in \mathscr{F}$ . This is our candidate.

We now need to show that $\mathfrak{M}$ satisfies (3). To do so, fix $S \subseteq X$.

*Claim* 4.1. $(x_\alpha)$ frequents all elements of $\{S \cap M : M \in \mathfrak{M}\}$ or all elements of $\left\{S^{\complement} \cap M : M \in \mathfrak{M}\right\}$.

Indeed, by contradiction assume that $(x_\alpha)$ doesn't frequent neither $S \cap M_1$ nor $S^{\complement} \cap M_2$ for some $M_{1,}M_2 \in \mathfrak{M}$. All the more the net doesn't frequent neither $S \cap M_1 \cap M_2$ nor $S^{\complement} \cap M_2 \cap M_1$, so neither their union which is $M_1 \cap M_2$; this contradicts (2).

The proof is now almost concluded. WLOG assume $(x_\alpha)$ frequents all elements of $\{S \cap M : M \in \mathfrak{M}\} =: \mathfrak{M}_1$ so that $\mathfrak{M}_1$ satisfies condition (1) and trivially also condition (2); hence $\mathfrak{M}_1 \in \mathscr{F}$ and then also $\mathfrak{M} \cup \mathfrak{M}_1 \in \mathscr{F}$, so that by maximality $\mathfrak{M} = \mathfrak{M} \cup \mathfrak{M}_1$ and then $\mathfrak{M}_1 \subseteq \mathfrak{M}$. In particular for every $M \in \mathfrak{M}$, $M \cap S \in \mathfrak{M}$, but $X \in \mathfrak{M}$ (otherwise $\{X\} \cup \mathfrak{M} \in \mathscr{F}$ would be greater) gives $S \in \mathfrak{M}$. Of course if in the claim we have that $(x_\alpha)$ frequents all elements of $\left\{S^{\complement} \cap M : M \in \mathfrak{M}\right\}$ the same argument gives $S^{\complement} \in \mathfrak{M}$. Hence $\forall S \subseteq X$ either $S \in \mathfrak{M}$ or $S^{\complement} \in \mathfrak{M}$, which gives (3) and concludes the argument. □

**Theorem 4.1.** (Existence of ultranets).

*Every net admits a subnet that is an ultranet.*

*Proof.* Let $(x_\alpha)_{\alpha \in I}$ be a net in $X$, let $\mathfrak{M}$ given by Lemma 4.2 and consider the set

$$J := \{(\alpha, M) \in I \times \mathfrak{M} : x_\alpha \in M\}$$

with the preorder $(\alpha_1, M_1) \preceq (\alpha_2, M_2)$ iff $\alpha_1 \leq \alpha_2$ and $M_1 \supseteq M_2$ and the mapping $\psi : J \longrightarrow I$ given by $\psi(\alpha, M) := \alpha$.

*Claim* 4.2. *$J$ is a directed set and $\psi$ defines a subnet.*

Indeed, given $(\alpha_1, M_1), (\alpha_2, M_2) \in J$ there is $\alpha_3 \in I$ with $\alpha_3 \geq \alpha_1$ and $\alpha_3 \geq \alpha_2$ and by properties (1) and (2) of $\mathfrak{M}$ we have that there is $\alpha_4 \geq \alpha_3$ with $x_{\alpha_4} \in M_1 \cap M_2$. This gives $(\alpha_4, M_1 \cap M_2) \in J$ and $(\alpha_4, M_1 \cap M_2) \succeq (\alpha_i, M_i)$, $i =$

$1, 2$; hence $J$ is a directed set. $\psi$ is clearly monotonic and onto (for each $\alpha \in I$, $(\alpha, X) \in J$), so defines a subnet.

*Claim* 4.3. *$(x_{\psi(\alpha,M)})$ is an ultranet.*

Indeed fixed $M_0 \in \mathfrak{M}$ there is $\alpha_0$ such that $x_{\alpha_0} \in M_0$, so for $(\alpha, M) \in J$ with $(\alpha, M) \succeq (\alpha_0, M_0)$ we have $x_{\psi(\alpha,M)} = x_\alpha \in M \subseteq M_0$; hence $(x_{\psi(\alpha,M)})$ is in $M_0$ ultimately. Now we use (3) to conclude: for each $S \subseteq X$ either $S$ or $S^{\complement}$ are in $\mathfrak{M}$, so the subnet is ultimately in one of them.

□

Now that we have ultranets we use them to give this characterization of compactness.

**Theorem 4.2.** (Ultranets and compactness).
*$X$ is compact iff every ultranet in $X$ converges in $X$.*

*Proof.* $\Longrightarrow$ Since $X$ is compact, by Theorem 3.2 every net must accumulate somewhere, so also every ultranet. But for them accumulation implies convergence.

$\Longleftarrow$ Pick any net in $X$ and by Theorem 4.1 extract a subnet which is an ultranet. This converges somewhere, so the net admits a convergent subnet and $X$ is compact. □

Finally we can now deduce as a trivial corollary:

**Corollary 4.1.** (Tychonoff Theorem).
*Any topological product of compact topological spaces is compact.*

*Proof.* Let $(x_\alpha)$ be a ultranet in the topological product, so that $(\pi x_\alpha)$ is an ultranet in one of the factors, hence converges since factors are compact. Thus the ultranet $(x_\alpha)$ converges coordinatewise, so converges in the topological product. □

We conclude answering another question which may rise: does Tychonoff Theorem hold for sequential compactness? I.e. is the topological product of sequentially compact topological spaces sequentially compact?

The answer is obviously yes in case of finite terms and using again the Cantor Diagonal Argument one easily gets that the answer is yes also for countably many factors.

If factors are more than countably many, then in general is false. We can prove this in at least two ways:

- by Banach-Alaoglu Theorem. If the product of sequentially compact spaces is sequentially compact, then $[-1, 1]^{B_X}$ is sequentially compact. Hence so is the closed subspace $B_{X^*} \subseteq [-1, 1]^{B_X}$ in the restriction of the product topology, which is the $w^*$ topology. Hence $B_{X^*}$ is sequentially $w^*$ compact, but we already mentioned that this is false. So briefly, sequential Tychonoff Theorem implies sequential Banach Alaoglu Theorem, which does not hold.

- with an explicit example. Consider $\{-1,1\}^{\mathbb{R}}$ i.e. functions from $\mathbb{R}$ to $\{-1,1\}$ with the pointwise convergence topology. We want to find $\{f_n\}$ sequence of functions such that no subsequence converges pointwise. Since $\#\{-1,1\}^{\mathbb{N}} = \mathfrak{c}$, we can put in bijection the set of all sequences of $\{-1,1\}$ with $\mathbb{R}$ and we use this as sequence of functions: $\forall \alpha \in \mathbb{R}$, $f_n(\alpha)$ is the n-th term of the sequence corresponding to $\alpha$. It's clear that no subsequence can converge at all points, since given any subsequence $n_k$ there exists of course a sequence $\{\delta_n\}$ with values in $\{-1,1\}$ such that $\{\delta_{n_k}\}$ oscillates and in the point corresponding to this sequence $\{f_{n_k}\}$ can not converge.

## 5. Tychonoff Theorem and the Axiom of Choice

The goal of this section is the proof of the aforementioned equivalence between the Axiom of Choice and the Tychonoff Theorem. Let us begin with the following simple equivalence.

**Lemma 5.1.** *The Axiom of Choice is equivalent to the fact that every surjective $g : Y \to X$ (for every pair of sets $X$ and $Y$) admits a right-inverse (that is: $f : X \to Y$ with $g \circ f = id_X$).*

*Proof.* Assume the existence of a right-inverse for every surjective map and let $X = \cup_{i\in I} X_i$ be a union of non-empty sets, indexed by a set $I$. Define the set $Y := \{(x,i) \in X \times I : x \in X_i\}$ and notice that from the assumption $X_i \neq \emptyset$ for every $i \in I$ it follows that the canonical projection $q : Y \to I$ is onto. Hence, our assumption yields a right-inverse $h : I \to Y$, that is $h(i) = (x_i, i)$ with $x_i \in X_i$. If $p : Y \to X$ denotes the canonical projection, then $p \circ h$ is the required choice function.

Conversely, let $g : Y \to I$ be onto; then for every $i \in I$ the set $X_i := g^{-1}(i)$ is non-empty and $X = \cup_{i\in I} X_i$. Hence the Axiom of Choice gives an $f : i \to X$ such that $f(i) \in X_i$, that is $g(f(i)) = i$ for every $i \in I$. □

**Theorem 5.1.** (Kelley).

*The Tychonoff Theorem implies the Axiom of Choice.*

*Proof.* Let $X, Y$ be sets and let $g : Y \to X$ be onto; we shall find a right-inverse for $g$. Notice that, fixed a finite subset $A$ of $X$, the set

$$F(A) := \{f : X \to Y : g(f(x)) = x \, \forall x \in A\} \subseteq Y^X$$

is non-empty and this assertion does not require the Axiom of Choice. Indeed, fixed $x_1 \in A$, $g^{-1}(x_1) \subseteq Y$ is not empty, so we can choose an element $f(x_1) \in g^{-1}(x_1)$. With finitely many choices we can build $f \in F(A)$.

Now observe that the family of all $F(A)$, as $A$ ranges in the family $\mathcal{F}$ of all finite subsets of $X$, has the finite intersection property; thus, if we could find a topology on $Y^X$ in which all $F(A)$ are closed and $Y^X$ is compact, then $\cap_{A\in\mathcal{F}} F(A)$ would be non-empty and every element in $\cap_{A\in\mathcal{F}} F(A)$ would be a right-inverse to $g$, thus concluding the proof. But since $F(A) = \cap_{x\in A} F(\{x\})$ it suffices to have that all the $F(\{x\}) = \{f : X \to Y : f(x) \in g^{-1}(x)\}$ are closed.

Consider the indiscrete topology $\tau_X$ on $X$ and let $\tau_Y := g^{-1}(\tau_X)$. Obviously, $g : (Y, \tau_Y) \to (X, \tau_X)$ is continuous and $(Y, \tau_Y)$ is compact (since $(X, \tau_X)$ is); hence by the Tychonoff Theorem $Y^X$, endowed with the product topology, is compact. Finally we have that $g^{-1}(x_0)$ is closed in $Y$ for every $x_0 \in X$, hence $\{f : X \to Y : f(x_0) \in g^{-1}(x_0)\}$ is closed in the product topology. $\square$

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