PDE Exercises

C.L. in Matematica e Matematica per le Applicazioni

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Chapter 1: Introduction to PDE

Exercise 1.1 - [Linearity and order of PDE]: see Exercise 1.5.1 of [E].

Exercise 1.2 - [Ellipticity of second order PDE] Given a second order PDE for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$; that is

$$F(x, u, Du, D^2u) = 0 \tag{1}$$

where $F = F(x, z, p, q) : \Gamma \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n) \to \mathbb{R}$ and $\mathrm{Sym}^2(\mathbb{R}^n)$ is the space of symmetric matrices with real entries. One says that equation (1) is *elliptic (for u) at the point* $x_0 \in \Omega$ if the matrix $\mathcal{M}(x_0) := D_qF(x_0, u(x_0), Du(x_0), D^2u(x_0))$ is positive or negative definite. Obviously, F must admit first order derivatives with respect to the variables $p \in \mathrm{Sym}^2(\mathbb{R}^n)$. The definition depends, in general, on the function u when the equation is quasilinear or fully nonlinear. For the following equations, determine conditions on $x, u(x), Du(x), D^2u(x)$ and the other function parameters for which the equation is elliptic. That is, determine suitable subsets Γ of $\mathcal{V}$ for which the equation is elliptic at each point of Ω.

- (Linear equations) Consider the general second order linear PDE with variable coefficients
$$\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u + \sum_{i=1}^{n} b_i(x)D_iu + cu - f(x) = 0$$
where $a_{ij} = a_{ji}, b_i$ and c are bounded functions on Ω. In particular, for which matrices $A = [a_{ij}(x_0)]$ is the equation elliptic at x_0? Does this depend on the other coefficients b_i, c or the forcing term f?
- (Minimal surface equation) Consider the following equation for a cartesian surface (the graph of a function u)
$$\mathrm{div}\left((1 + |Du|^2)^{-1/2}Du\right) = 0$$
(**Hint:** begin with the case $n = 2$.
- (Potential flow) Consider the following equation
$$\left(1 - \frac{u_x^2}{c^2}\right)u_{xx} - 2\frac{u_xu_y}{c^2}u_{xy} + \left(1 - \frac{u_y^2}{c^2}\right)u_{yy} = 0$$
where u is the *velocity potential* which satisfies $Du = (u_x, u_y)$ the velocity vector for the flow and $c = c(|Du|)$ is *the local sound speed* which is a function of the flow speed $|Du|$.
- (Monge-Ampère) Consider the equation
$$\det\left(D^2u\right) = 0$$
(**Hint:** begin with the case $n = 2$)
Repeat the exercise for the more general equation of *Monge-Ampère type*
$$\det\left(D^2u\right) + f(x, u, Du) = 0$$

where $f = f(x, z, p) : \mathcal{W} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

Exercise 1.3 - [Degenerate ellipticity of second order PDE] One says that equation (1) from Exercise 1.2 is *degenerate elliptic (on* $\Gamma \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n)$*)* if the function F is monotone in q; that is

$$F(x, z, p, A) \leq F(x, z, p, B) \quad \text{if } A \leq B \quad \forall\, (x, z, p),$$

or

$$F(x, z, p, A) \geq F(x, z, p, B) \quad \text{if } A \leq B \quad \forall\, (x, z, p),$$

where two matrices $A, B \in \mathrm{Sym}^2(\mathbb{R}^n)$ satisfy $A \leq B$ if the quadratic form associated to A is dominated by that of B; that is

$$\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle \quad \forall\, \xi \in \mathbb{R}^n.$$

Study the question of degenerate ellipticity for the examples in Exercise 1.2.

Exercise 1.4 - [Multi-indices and the formulas of Leibniz and Taylor]: see Exercises 1.5.2, 1.5.3, 1.5.4 and 1.5.5 of [E].

Exercise 1.5 - [Basic notions]: Look at the appendices A, B, C, D, E of [E] for background notions that will be useful throughout the course. For now, we do **NOT** need D.5, D.6 e E.5.

References

[E] - Evans, L.C. - *Partial Differential Equations, Second Edition*, Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., Providence, RI, 2010.