PDE Exercises

C.L. in Matematica e Matematica per le Applicazioni

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Chapter 2: Representation formulas for solutions

Exercise 2.1 - [Transport equations and the method of characteristics]: Read §2.1 of [P2].

Exercise 2.2 - [Transport equation with zero order term]: Exercise 2.5.1 of [E].

Exercise 2.3 - [Method of characteristics for linear equations of first order]: Exercise 3.5.4 of [E].

Exercise 2.4 - [Null bicharacteristics and characteristics for second order linear operators]: Given a general linear partial differential operator of second order

$$Pu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} b_i(x) D_i u + cu$$

with smooth (C^∞) coefficients $a_{ij} = a_{ji}, b_i, c$, one defines the *principal symbol* of P as

$$\sigma(x, \xi) = \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j.$$

The *null bicharacteristic of P through* (x_0, ξ_0) is defined as the solution curve $\Gamma(s) = (x(s), \xi(s))$ which solves the Hamiltonian system

$$\dot{x} = D_\xi \sigma$$
$$\dot{\xi} = -D_\xi \sigma$$

with the initial (null) condition

$$\sigma(x_0, \xi_0) = 0.$$

The curve $\gamma(s) = x(s)$ is then called a *characteristic of P*.

- **a)** The *wave operator* on $\mathbb{R}^n \times \mathbb{R}$ with coordinates (x, t) has the form $P = D_t^2 - \sum_{j=1}^n D_{x_j}^2$ and so the principal symbol is

  $$\sigma(x, t, \xi, \tau) = \tau^2 - |\xi|^2$$

  and the null initial condition at $p_0 = (x_0, t_0, \xi_0, \tau_0)$ is

  $$\tau_0^2 - |\xi_0|^2 = 0$$

  Find the null bicharacteristic passing through p_0. Show that all characteristics through (x_0, t_0) are straight lines living on the *light cone*

  $$\Sigma_0 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \ |t - t_0| = |x - x_0|\}.$$

- **b)** The *Tricomi operator* on $\mathbb{R}^2$ with coordinates (x, y) has the form $P = yD_x^2 + D_y^2$ and so the principal symbol is

  $$\sigma(x, y, \xi, \eta) = y\xi^2 + \eta^2$$

  and the null initial condition at $p_0 = (x_0, y_0, \xi_0, \eta_0)$ is

  $$y_0 \xi_0^2 + \eta_0^2 = 0, \quad y_0 \leq 0.$$

Find the null bicharacteristic passing through p_0. Show that all maximally extended characteristics through (x_0, y_0) reach the y-axis and form a cusp there.

Exercise 2.5 - [Invariances for the Laplacian]: Exercises 2.5.2 and 2.5.11 of [E] for the invariance with respect to rotations and inversions. Verify also the other invariances stated in class: translations, dilations and inversion with respect to spheres (see also the notes [P1]).

Exercise 2.6 - [Estimates on the fundamental solution of the Laplacian]: For $\Phi(x) = \Gamma(|x|)$ the fundamental solution of Δ in $\mathbb{R}^n$ where

$$\Gamma(|x|) = \begin{cases} \dfrac{1}{2\pi} \log(|x|) & n = 2 \\ \\ \dfrac{1}{n(2-n)\omega_n}|x|^{2-n} & n \geq 3 \end{cases}$$

find and explicit expreession for the derivatives $D_i\Phi, D_{ij}\Phi$ when $x \neq 0$ and use them to verify Proposition 2.2.1 in the class notes [P2]:

$$|D_i\Phi(x)| \leq \frac{1}{n\omega_n}|x|^{1-n}; \ \ |D_{ij}\Phi(x)| \leq \frac{1}{\omega_n}|x|^{-n}.$$

Generalize the result to find

$$|D^\alpha\Phi(x)| \leq C(n, \alpha)|x|^{2-n-|\alpha|}$$

with $\alpha \in \mathbb{N}_0^n$ a multi-index.

Exercise 2.7 - [Green's second identity]: Check the claim made in class (Proposition 2.2.3): if $u, v \in C^2(\overline{\Omega})$ with $\partial\Omega \in C^1$ then

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) dS$$

where ν is the external unit normal vector.

Exercise 2.8 - [Green's representation]: Check the claim made in class (Theorem 2.2.2): with $B_\epsilon(y)$ a small ball about the singularity of $\Phi(\cdot - y)$, one has

$$-\frac{\partial\Phi}{\partial\nu}(\cdot - y)_{|\partial B_\epsilon(y)} = \frac{1}{|\partial B_\epsilon(y)|}$$

where $|\partial B_\epsilon(y)|$ is the $(n-1$ dimensional) measure of the sphere and ν is the external unit normal (with respect to $\Omega \setminus \overline{B}_\epsilon(y)$.

Exercise 2.9 - [Symmetry of the Green's function]: Verify the claim of Proposition 2.2.4 about the symmetry of the *corrector function for the ball* $B_R = B_R(0)$; that is, $\varphi(x; y) = \varphi(y; x)$. In this way, one finds the symmetry of the Green's function $G(x; y) = \Gamma(|x-y| + \varphi(x; y)$ which was mentioned in the sketch of the proof of Theorem 2.2.5 done in class.

Exercise 2.10 - [Solving Dirichlet's Problem on Balls]: Check the rest of the proof of Theorem 2.2.5 which was sketched in class (see [P2]).

Exercise 2.11 - [Poisson's integral formula]: Check the claim made in class (Theorem 2.2.4) on the normal derivative of the Green's function:

$$-\frac{\partial G}{\partial\nu}(x; y) = \frac{R^2 - |y|^2}{n\omega_n R}\frac{1}{|x-y|^n}, \ \ x \in \partial B_R$$

.

Exercise 2.12 - [Compatibility conditions for the Neumann problem]: Consider the Neumann problem on a bounded domain Ω with C^1 boundary; that is

$$\text{(NP)} \quad \begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \partial\Omega, \end{cases}$$

with f, h given continuous functions. Prove the following statements about solutions $u \in C^2(\overline{\Omega})$ to (NP):

(CC1) If $h = 0$ then $\int_\Omega f(x)\, dx = 0$ (i.e. f must have average value zero on Ω);

(CC2) If $f = 0$ then $\int_{\partial\Omega} h\, dS(x) = 0$ (i.e. h must have average value zero on $\partial\Omega$)

Exercise 2.13 - [Representation formula involving the Neumann function]: Consider the following representation formula for $u \in C^2(\overline{\Omega})$ with Ω a bounded domain with C^1 boundary:

$$(*) \quad u(y) = \int_\Omega N(x;y)\Delta u(x)\, dx - \int_{\partial\Omega} N(x:y)\frac{\partial u}{\partial \nu}\, dS(x) + \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u(x)\, dS(x)$$

where $|\partial\Omega|$ is the surface measure of $\partial\Omega$ and $N(x;y) = \Phi(x-y) - \psi(x;y)$ is the *Neumann function* with $\psi(\cdot;y)$ defined as the solution of

$$(**) \quad \begin{cases} \Delta_x \psi(\cdot;y) = 0 & \text{in } \Omega \\ \dfrac{\partial \psi}{\partial \nu}(\cdot;y) = \dfrac{\partial \Phi}{\partial \nu}(\cdot - y) - \dfrac{1}{|\partial\Omega|} & \text{on } \partial\Omega, \end{cases} \qquad \forall y \in \Omega.$$

a) Show that for each $y \in \Omega$ the compatibility condition (CC2) of Exercise 2.12 holds for $\psi(\cdot;y)$ and hence a solution of $(**)$ can exist. On the other hand, without the term $-1/|\partial\Omega|$ the corresponding system would not have a classical solution. **Hint:** Use $u \equiv 1$ in Green's Representation Formula to verify that $\int_{\partial\Omega} \frac{\partial \Phi}{\partial x}(x-y)\, dS(x) = 1$.

(b) Show that $(*)$ holds for each $y \in \Omega$ if there exists a solution $\psi(\cdot;u)$ of $(**)$ for each $y \in \Omega$.

Exercise 2.14 - [Divergence form equations, Green's identities and uniqueness]: Let Ω be a bounded domain with C^1 boundary. Consider the following operator in *divergence form*

$$Lu := \operatorname{div}[ADu] = \sum_{i,j=1}^n D_j(a_i j D_i u),$$

where the coefficients $a_{ij} \in C^1(\overline{\Omega})$ are real valued and satisfy the symmetry condition $a_{ij} = a_{ji}$ for each $i, j = 1, \dots n$; that is, the matrix valued function A is symmetric.

a) For $u, v \in C^2(\overline{\Omega})$ establish the following *Green's identities*

$$\int_\Omega vLu\, dx = -\int_\Omega \langle Dv, ADu\rangle\, dx + \int_{\partial\Omega} vD_\nu u\, dS$$

$$\int_\Omega [vLu - uLv]\, dx = \int_{\partial\Omega} [vD_\nu u - uD_\nu v]\, dS$$

where ν is the external unit normal field on $\partial\Omega$ and $D_\nu u := \langle ADu, \nu\rangle$ defines the *conormal derivative* operator on $\partial\Omega$.

(b) If in addition A is *non-negative definite*; that is, $\langle A(x)\xi, \xi\rangle \geq 0$ for each $x \in \Omega, \xi \in \mathbb{R}^n$, then the square root $A^{1/2}$ of A is well defined symmetric matrix valued function which is continuous on Ω. Find and justify an additional condition on A such that $C^2(\overline{\Omega})$ solutions of the Dirichet and Neumann problems

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on} \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} Lu = f & \text{in } \Omega \\ D_\nu u = h & \text{on} \partial\Omega, \end{cases}$$

will be unique and unique up to an additive constant respectively.

Exercise 2.15 - [Subharmonic functions and mean value inequalities]: see Exercise 2.5.5 of [E].

Exercise 2.16 - [Pointwise estimates using the maximum principle]: see Exercise 2.5.6 of [E].

Exercise 2.17 - [Estimates from the mean value property]: Let u be harmonic in Ω. For each $B_r(x_0) \subset\subset \Omega$ and for each $\alpha \in \mathbb{N}_0^n$, there exists $C = C(n, |\alpha|) > 0$ such that

$$|D^\alpha u(x_0)| \leq Cr^{-n-k}\|u\|_{L^1(B_r(x_0))}.$$

(see Theorem 2.2.7 of [E])

Exercise 2.18 - [Liouville's Theorem]: Let u be harmonic in $\mathbb{R}^n$. If u is bounded, then u is constant (see Theorem 2.2.8 of [E]).

Exercise 2.19 - [Harnack's inequality]:

- Let u be harmonic and $\underline{\text{non negative}}$ in Ω. For each domain subdomain $\mathcal{V} \subset\subset \Omega$, there exists $C = C(\mathcal{V}) > 0$ such that
$$\sup_{\mathcal{V}} u \leq C \inf_{\mathcal{V}} u$$
(see Theorem 2.2.11 of [E]).
- See Exercise 2.5.7 of [E] for an explicit version by way of the Poisson integral formula.

Exercise 2.20 - [The Dirichlet problem on half-spaces]: see Exercise 2.5.9 of [E].

Exercise 2.21 - [Schwarz reflection principle]: see Exercise 2.5.10 of [E].

Exercise 2.22 - [The heat equation]: Read §2.3 of [P2].

Exercise 2.23 - [Invariances for the heat equation]: see Exercise 2.5.12 of [E]

Exercise 2.24 - [Elementary solutions to the heat equation]: Check Proposition 2.3.1 of [P2]. In particular, verify that u defined in (2.3.3) has the properties claimed and read the derivation given in the notes.

Exercise 2.25 - [Solving the Cauchy problem for the heat equation]: Study the proof of Theorem 2.3.1 of [P2].

Exercise 2.26 - [Non homogeneous heat equation]: Read the proof of Theorem 2.3.2 of [E] (which corresponds to Theorem 2.3.2 of [P2]).

Exercise 2.27 - [Non homogeneous heat equation with zero order term]: see Exercise 2.5.14 of [E].

Exercise 2.28 - [Mean value property for the heat equation]:

- Read the proof of Theorem 2.3.3 of [E].
- Check the key claim made at the end of the proof; that is,

$$\frac{1}{r^n}\int_{E(r)}\frac{|y|^2}{s^2}\,dsdy=\int_{E(1)}\frac{|y|^2}{s^2}\,dsdy=4$$

Exercise 2.29 - [Uniqueness by the energy method]: Using the multiplier u, show that there is at most one solution $u\in C^2_1(\overline{\Omega}_T)$ of the Cauchy-Dirichlet problem for the non homogeneous heat equation (see Theorem 2.3.10 of [E]).

Exercise 2.30 - [Regularity and estimates for the heat equation]: Read the proofs of the Theorems 2.3.8 and 2.3.9 of [E].

Exercise 2.31 - [Backwards uniqueness for the heat equation]: Read the proof of Theorem 2.3.11 of [E].

Exercise 2.32 - [The wave equation]: Read §2.4 of [P2].

Exercise 2.33 - [D'Alembert's formula revisited]: see Exercise 2.5.19 of [E].

Exercise 2.34 - [Stokes' rule]: see Exercise 2.5.18 of [E]

Exercise 2.35 - [Method of reflections]: Check the claims made in Observation 2.4.2 of the class notes [P2]; that is that the formula

$$u(x,t)=\begin{cases}\frac{1}{2}[g(x+t)+g(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}h(y)\,dy & x\geq t\geq 0\\[2ex] \frac{1}{2}[g(t+x)-g(t-x)]+\frac{1}{2}\int_{t-x}^{t+x}h(y)\,dy & 0\leq x\leq t\end{cases}$$

gives a solution of the problem

$$u_{tt}-u_{xx}=0\ \text{ in }\ (0,+\infty)\times(0,+\infty)$$

$$u(x,0)=g(x);u_t(x,0)=h(x)\ \text{ per }\ x\in(0,+\infty)$$

$$u(0,t)=0\ \text{ per }\ t\in(0,+\infty)$$

Exercise 2.36 - [Method of spherical means]: Prove Lemma 2.4.1 of [P2] concerning the reduction to the Euler-Poisson-Darboux equation (see Lemma 2.4.1 of [E]).

Exercise 2.37 - [Poisson's formula]: Finish the proof of Theorem 2.4.1 of [P2]; (see pp. 73-74 of [E]).

Exercise 2.38 - [Duhamel's principle]: Prove Theorem 2.4.4 of [E] in the cases $n=1,2,3$.

Exercise 2.39 - [Energy methods]: Study the proofs of Theorems 2.4.3 and 2.4.4 of [P2].

Exercise 2.40 - [Equipartition of energy]: see Exercise 2.5.24 of [E].

References

[E] - Evans, L.C. - *Partial Differential Equations, Second Edition*, Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., Providence, RI, 2010.

[P1] Payne, K.R. - *Funzioni Armoniche: Un Primo Assaggio*, (2006), available at the address `http://www.mat.unimi.it/users/payne/anIIIfunzarm05-06.pdf`

[P2] Payne, K.R. - *Equazioni alle Derivate Parziali: Appunti del Corso*, (2015), available at the address `http://www.mat.unimi.it/users/payne/PDE_AA14_15.pdf`