PDE Exercises

C.L. in Matematica e Matematica per le Applicazioni

Prof. Kevin R. Payne

Chapter 4: Linear elliptic PDE of second order

Exercise 4.1 - [Laplace's equation with potential]: see Exercise 6.6.1 of [E].

Exercise 4.2 - [Characteriziation of $H^{-1}(\Omega)$]: Read the proof of Theorem 5.9.1 of [E] and illustrate part i) of the theorem for the following linear functionals, where Ω is a bounded open set in $\mathbb{R}^n$. That is, identify the functions $f_0, f_1, \dots f_n$ in $L^2(\Omega)$ which appear in the statement of the theorem.

a) For $f \in L^2(\Omega)$ fixed, define l_f by

$$\langle l_f, v\rangle := \int_\Omega fv\,dx = (f,v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

b) For L a uniformly elliptic operator of second order, in divergence form, with bounded coefficients and $u \in H_0^1(\Omega)$, define l_{Lu} by

$$\langle l_{Lu}, v\rangle := B[u,v], \quad \forall v \in H_0^1(\Omega),$$

where $B[u,v]$ is the bilinear form associated to L.

Exercise 4.3 - [Continuity of L_μ^{-1}]: Finish the proof of Corollary 4.2.2 by showing the continuity of $L_\mu^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$.

Exercise 4.4 - [Formal adjoint of L]: Verify the claim made after Definition 4.2.2 on the form of L^*v if $Lu = -\sum_{i,j=1}^n D_j(a_{ij}D_iu) + \sum_{i=1}^n b_iD_iu + cu$

Exercise 4.5 - [Fredholm alternative]: Complete the proof of Theorem 4.2.2 using Lemma 4.2.5 and consulting Theorem 6.2.4 of [E] (and also Theorem D.5.5 if necessary).

Exercise 4.6 - [A variant of Theorem 4.2.1 on existence]: see Exercise 6.6.2 of [E]

Exercise 4.7 - [The Dirichlet problem with homogeneous boundary data]: Let L be as in §4.2 of the class notes [P]; that is, a uniformly elliptic operator in divergence form with L^∞ coefficients ($a_{ij} = a_{ji}$). Define $\mu \in \mathbb{R}$ to be a *good parameter* if for each $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ to the Dirichlet problem

$$Lu + \mu u = f \ \text{ in } \ \Omega$$

$$u = 0 \ \text{ su } \ \partial\Omega$$

and define μ to be a *bad parameter* if it is not a good parameter. Give a complete description of the good and bad cases. In particular, answer the following questions. If μ is bad can there be uniqueness? If μ is bad can there be existence for every f? If not, for which f? Describe completely the structure of the set of bad μ and hence of the good μ.

Exercise 4.8 - [The Dirichlet problem with nonhomogeneous boundary data]: Formulate and prove a generalization of Theorem 4.2.1 for the problem

$$Lu + \mu u = f \ \text{ in } \ \Omega$$

$$u = g \ \text{ su } \ \partial\Omega$$

with f and g given elements of a suitable Hilbert space.

Exercise 4.9 - [The Neumann problem for the Laplacian]: see Exercises 6.6.4 and 6.6.10 of [E].

Exercise 4.10 - [Other boundary conditions for the Laplacian]: see Exercises 6.6.5 and 6.6.6 of[E].

Exercise 4.11 - [The biharmonic equation]: see Exercise 6.6.3 of [E].

Exercise 4.12 - [Interior regularity $H^2(\Omega)$]: Complete step 7 of the proof of Theorem 4.3.1 (consulting Theorem 6.3.1 of [E]).

Exercise 4.13 - [Higher regularity up to the boundary]: Get a better idea of the proofs of Theorems 4.3.2, 4.3.3, 4.3.4 by consulting Theorems 6.3.2, 6.3.3, 6.3.4, 6.3.5 of [E].

Exercise 4.14 - [Internal regularity $H^2(\Omega)$ for a semilinear equation]: see Exercise 6.6.7 of [E].

Exercise 4.15 - [$W^{1,\infty}$ estimates using the maximum principle]: see Exercise 6.6.8 of [E].

Exercise 4.16 - [Barrier functions and gradient estimates]: see Exercise 6.6.9 of [E].

Exercise 4.17 - [Weak subsolutions to linear equations]: see Exercise 6.6.11 of [E].

Exercise 4.18 - [Weak comparison principle for classical solutions of linear equations]: see Exercise 6.6.12 of [E]. Notice that there is no hypothesis on the sign of the coefficient c.

Exercise 4.19 - [Eigenvalues and eigenfunctions]: Complete the proofs of Theorems 4.5.1 and 4.5.2 by consulting Theorems 6.5.1 and 6.5.2 of [E] (and also section D.6 of [E] if necessary).

Exercise 4.20 - [Principle eigenvalue for non-symmetric operators]: Read the statement and the proof of Theorem 6.5.3 in [E]. For a deeper study, see also [BNV].

Exercise 4.21 - [Minimax principle for λ_1]: see Exercises 6.6.13 and 6.6.14 of [E].

Exercise 4.22 - [Eigenvalues and domain variations]: see Exercise 6.6.15 of [E].

Exercise 4.23 - [Maximum principles for classical solutions]: Read section 4.4.1 of [P] and think about Theorems 4.41 and 4.4.2.

Referenze

[BNV] - Berestycki, H., Nirenberg, L., e Varadhan, S.R.S. - *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), 47–92.

[E] - Evans, L.C. - *Partial Differential Equations, Second Edition*, Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., Providence, RI, 2010.

[P] Payne, K.R. - *Equazioni alle Derivate Parziali: Appunti del Corso*, (2015), available at the address http://www.mat.unimi.it/users/payne/PDE_AA14_15.pdf