

### Chap 3. Hilbert Spaces

LEZ 16  
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Point: Study complete inner product spaces and the bounded linear operators on them

#### 3.1 Definitions and elementary properties

##### Inner product spaces

Def1: Given a vector space  $H$  over  $\mathbb{R}$ , an inner product on  $H$

is a map  $(\cdot, \cdot): H \times H \rightarrow \mathbb{R}$  s.t.

(i)  $(x, x) \geq 0 \quad \forall x \in H$  and  $(x, x) = 0 \Rightarrow x = 0$  (positivity)

(ii)  $(x, y) = (y, x) \quad \forall x, y \in H$  (symmetry)

(iii)  $\begin{cases} (\cdot, y): H \rightarrow \mathbb{R} & \text{linear} \\ (\cdot, \cdot): H \rightarrow \mathbb{R} & \forall x \in H \end{cases}$  (a) (bilinearity)  
(b)

N.B. ① Sometimes we write  $(\cdot, \cdot)_H$ ; especially when there are more than one inner products in play

② with (ii), we need only one of the linearity statements in (iii)

③ We call  $(H, (\cdot, \cdot)_H)$  an inner product space

Def2: Given a vector space  $H$  over  $\mathbb{C}$ , an inner product on  $H$  is a map  $(\cdot, \cdot): H \times H \rightarrow \mathbb{C}$  s.t.

(i) positive definite (as before)  $[(x, x) \in \mathbb{R}]$

(ii) skew symmetry  $(x, y) = \overline{(y, x)} \quad \forall x, y \in H$

(iii) sequilinear  $\begin{cases} (\cdot, y) \text{ linear } \forall y \in H \text{ fixed} \\ (x, \cdot) \text{ skew linear } \forall x \in H \text{ fixed} \end{cases}$  (a)  
(b)

where (b):  $\begin{cases} (x, y+z) = (x, y) + (x, z) \\ \forall x, y, z \in H \end{cases}$

$\begin{cases} (x, \lambda y) = \lambda(x, y) \\ \forall x, y \in H, \forall \lambda \in \mathbb{C} \end{cases}$

N.B. in (b)

$$\circ (x, \lambda y) = \overline{(\lambda y, x)} = \overline{\lambda(y, x)} = \bar{\lambda}(\bar{y}, x) = \bar{\lambda}(x, y)$$

(iii) (iv)(a)

$$\bullet (x, y+z) = (\overline{y+z}, x) = \overline{(y, x) + (z, x)} = (\overline{y}, x) + (\overline{z}, x)$$

## Novus:

$$= (x, y) + (x, z)$$

Def 3: If  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space, then

one has a norm  $\|\cdot\|: H \rightarrow \mathbb{R}$  defined by

$$\|x\| = (x, x)^{\frac{1}{2}}, \quad \forall x \in H$$

$$(N1) \quad \|x\| \geq 0 \text{ with } \|x\|=0 \iff x=0$$

• Comes from (ii) :  $(\cdot, \cdot)$  is positive definite

$$(N2) \| \lambda x \| = \| \lambda \| \| x \|$$

• Comes from (iii) linear properties

$$\|\lambda x\|^2 = (\lambda x, \lambda x) = \lambda(x, \lambda x) = \lambda \bar{\lambda} (x, x) = |\lambda|^2 \|x\|^2$$

$$(N3) \quad \|x+y\| \leq \|x\| + \|y\|$$

• This is a consequence of the Cauchy-Schwarz inequality

Lemma 1: (Schwartz Ineq) In an inner product space  $(H, (\cdot, \cdot))$

$$(S1) \quad |(x,y)| < |(x_0,y_0)| \quad \forall x,y \in H$$

with " $\equiv$ " iff  $y = 0$  or  $x = \alpha y$  for some  $\alpha$  (in  $\mathbb{R}, \mathbb{C}$ )

proof: (Complex case)

1. obvious if  $y=0$  with " $=$ "

2 If  $y \neq 0$ , then  $\forall \lambda \in \mathbb{Q}$

$$\begin{aligned}
 0 \leq \|x + \lambda y\|^2 &= (x + \lambda y, x + \lambda y) \\
 &= (x, x) + (x, \lambda y) + (\lambda y, x) + (\lambda y, \lambda y) \\
 &= \|x\|^2 + (\overline{\lambda y}, x) + (\lambda y, x) + |\lambda|^2 \|y\|^2 \\
 &= \|x\|^2 + 2\operatorname{Re}(\lambda(x, y)) + |\lambda|^2 \|y\|^2
 \end{aligned}$$

Pick  $\lambda = -\frac{(x,y)}{\|y\|^2}$  to find (calculation)

$$0 \leq \|x + \lambda y\|^2 = \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2}$$

$$\Rightarrow \frac{|(x,y)|^2}{\|y\|^2} \leq \|x\|^2 \quad (\text{CSI})$$

with " $=$ " if  $x + \lambda y = 0$  some  $\lambda \in \mathbb{C}$ , i.e.  $x = -\lambda y = \lambda y$ .



Thm 1: If  $(H, \langle \cdot, \cdot \rangle)$  is an inner product space, then

$$\|x\| := \sqrt{\langle x, x \rangle}$$

is a norm on  $H$

Proof: just need (N3) triangle in eq.

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2\operatorname{Re}(x,y) + \|y\|^2 \\ &\leq \|x\|^2 + 2|(x,y)| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \end{aligned}$$



Cor 1: In an inner product space,

$$\|x\| = \max_{\|y\|=1} |(x,y)|$$

1. wlog  $x \neq 0$  (obvious if  $x=0$ )

Proof: 2. By (CSI)  $|(x,y)| \leq \|x\|\|y\| \leq \|x\| \|y\| \leq \|y\| \leq 1$

$$\text{so } \sup_{\|y\|=1} |(x,y)| \leq \|x\|$$

3. For  $y = \frac{x}{\|x\|}$  where  $\|y\|=1$  leave  $\leq$  in (CSI)

$$|(x, \frac{x}{\|x\|})| = \|x\| \text{ so } \|x\| \text{ is a max.}$$



Question: Given a normed vector space  $(H, \|\cdot\|)$ , when is there an inner product  $\langle \cdot, \cdot \rangle$  such that  $\|x\| = \sqrt{\langle x, x \rangle}$ ?

Answer: The norm must satisfy the parallelogram law

$$(\text{PL}) \quad \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in H$$

- Exercises:
- 1) If  $(H, (\cdot, \cdot))$  is an inner product space, then  
(PL) holds [necessary condition]
  - 2) (PL) is a sufficient condition of  $H \cdot H$  for the existence  
Hilbert spaces of  $(\cdot, \cdot)$

Def 4: A Hilbert space  $H$  is a complete normed vector space with inner product  $(\cdot, \cdot)$  compatible with the norm  $\|\cdot\|$ ; that is,  $\|x\| = (x, x)^{1/2}$

Examples:

- ①  $\mathbb{R}^n, \mathbb{C}^n$  with standard inner products

$$(x, y)_{\mathbb{R}^n} = \sum_{j=1}^n x_j y_j \quad (z, w)_{\mathbb{C}^n} = \sum_{j=1}^n z_j \bar{w}_j$$

- ② Fix  $A = [a_{ij}]_{n \times n} \in S(n)$  [symmetric, real,  $n \times n$  matrix]

$$(x, y)_A = (Ax, y)_{\mathbb{R}^n} \text{ gives a scalar product on } \mathbb{R}^n \text{ if } A \text{ is positive definite}$$

(A = Id is the euclidean case)

N.B. on  $\mathbb{C}^n$  use  $A$  s.t.  $A = A^*$   $a_{ij} = \bar{a}_{ji}$  (Hermitian symmetric)

- ③  $L^2(\mathbb{R}^n) = \{x = \{x_j\}_{j=1}^{+\infty} : x_j \in \mathbb{R}, \sum_{j=1}^{+\infty} x_j^2 < +\infty\}$

$$(x, y)_{L^2(\mathbb{R}^n)} = \sum_{j=1}^{+\infty} x_j y_j$$

- ④  $L^2(E)$  with  $E \in M(\mathbb{R}^n), |E| > 0$

$$(f, g)_{L^2(E)} = \int_E f g \, dx$$

- ⑤  $L^2(X, m(x), \mu)$  with  $(f, g) = \int_X f g \, d\mu$

N.B. in complex cases  $\int_X f \bar{g} \, d\mu$

Exercise: Show that the Banach space  $L^p(E)$  is Hilbert iff  $p=2$  (validity of (PL))

### 3.2 Orthogonality and the Projection theorem

Goal: Examine the geometry given by  $\langle \cdot, \cdot \rangle$  and use it to solve some optimization problems of finding the closest elements to a fixed subset

Def1: In an inner product space  $(H, \langle \cdot, \cdot \rangle)$ , two elements  $x, y \in H$  are orthogonal if  $\langle x, y \rangle = 0$  and we write  $x \perp y$

Prop1: (a)  $x \perp y \Leftrightarrow y \perp x$

(b)  $x \perp y \forall y \in H \Leftrightarrow x = 0$

(c)  $x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$  (Pythagoras)

proof: (a) (Skew) symmetry of  $\langle \cdot, \cdot \rangle$

(b) positive definiteness of  $\langle \cdot, \cdot \rangle$  + "linearity"

$$(c) \|x+y\|^2 = \|x\|^2 + \langle x, y \rangle^2 + \langle y, x \rangle^2 + \|y\|^2$$

N.B. If  $H$  real: Pythagoras  $\Rightarrow x \perp y$

$H$  complex:  $\nrightarrow \Rightarrow \operatorname{Re} \langle x, y \rangle = 0$

Def2 For  $M \subset H$  non-empty, the orthogonal space to  $M$  is defined by

$$M^\perp = \{ y \in H : \langle y, x \rangle = 0 \quad \forall x \in M \}$$

Prop2: Let  $M \subset H$ ,  $M \neq \emptyset$  in  $H$  Hilbert

(a)  $M^\perp$  is a closed linear subspace of  $H$

(b)  $(M^\perp)^\perp = M$  and is the closed subspace generated by  $M$   
(the smallest closed subspace containing  $M$ )

proof: (a)  $M^\perp$  is a closed subspace:

1.  $y_1, y_2 \in M^\perp \Rightarrow \forall x \in M \quad \langle y_1 + y_2, x \rangle = \langle y_1, x \rangle + \langle y_2, x \rangle = 0 + 0 = 0$   
 $\Rightarrow y_1 + y_2 \in M^\perp$

2.  $y \in M^\perp, \lambda \text{ scalar} \Rightarrow \forall x \in M \quad \langle \lambda y, x \rangle = \lambda \langle y, x \rangle = \lambda(0) = 0$   
 $(\mathbb{C}, \mathbb{R}) \Rightarrow \lambda y \in M^\perp$

3. let  $\{y_j\}_{j \in \mathbb{N}} \subset M^\perp$  and  $y_j \rightarrow y$  in  $H$  norm as  $j \rightarrow \infty$

- $\forall x \in H \quad (y_j, x) \rightarrow (y, x)$

$$|(y_j, y, x)| \leq \|x\| \|y_j - y\| \rightarrow 0$$

- $\forall x \in M \quad (y, x) = \lim_{j \rightarrow \infty} (y_j, x) = 0$   
 $\Rightarrow y \in M^\perp$  (closed)

(b)  $M^\perp \neq \emptyset$  since  $0 \in M^\perp$

$\Rightarrow (M^\perp)^\perp$  is a closed subspace of  $H$

- $M \subset (M^\perp)^\perp$ ? Let  $y \in M^\perp$  be arbitrary

for each  $x \in M \quad (x, y) = 0 \quad \text{so} \quad x \perp M^\perp$

Hence  $(M^\perp)^\perp$  contains  $x$

(minimality?) 

Exercise:  $(M^\perp)^\perp = X$  for each  $X$  closed subspace of  $H$  containing  $M$

Remark: We obtain much more if

(a)  $M = K$  closed, convex

(b)  $M = X$  closed, subspace

Thm 1: Let  $K \subset H$  be a closed, convex, non empty subset of  $H$  Hilbert-

Then  $\forall x_0 \in H \quad \exists! y_0 \in K$  minimizing the distance to  $K$ ;  
 that is,

$$\|x_0 - y_0\| = \inf_{y \in K} \|x_0 - y\| \doteq \text{dist}(x_0, K)$$

N.B. We call  $y_0$  the projection of  $x_0$  onto  $K$ ;  $y_0 = P_K x_0$

and it gives the best approximation of  $x_0$  by an element of  $K$ .

In particular,  $P_K x_0 = x_0$  if  $x_0 \in K$ .

Proof:  $\text{dist}(x_0, K) \geq 0$  so  $\inf_{y \in K} \|x_0 - y\|$  exists and is  $\geq 0$

1. Pick a minimizing sequence  $\{y_n\}_{n \in \mathbb{N}}^{\subset K}$  for  $\text{dist}(x_0, K) \stackrel{100}{=} d$

$$\text{i.e. } d_n = \text{dist}(x_0, y_n) \rightarrow d$$

$$[\|x_0 - y_n\|]$$

2. Claim:  $\{y_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence w.r.t.  $\|\cdot\|$

• Apply (PL) to i.  $x = \frac{x_0 - y_n}{2}, y = \frac{x_0 - y_m}{2}$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (\text{PL})$$

$$\left\| x_0 - \frac{y_n + y_m}{2} \right\|^2 + \left\| \frac{y_n - y_m}{2} \right\|^2 = 2\left(\frac{d_n^2}{4}\right) + 2\left(\frac{d_m^2}{4}\right)$$

$$(*) \quad \left\| x_0 - \frac{y_n + y_m}{2} \right\|^2 + \frac{1}{4}\|y_n - y_m\|^2 = \frac{1}{2}(d_n^2 + d_m^2)$$

• K convex  $\Rightarrow \frac{y_n + y_m}{2} \in K$

$$\Rightarrow \left\| x_0 - \frac{y_n + y_m}{2} \right\|^2 \geq d^2$$

$$(*) \quad \|y_n - y_m\|^2 \leq 4 \left[ \frac{1}{2}(d_n^2 + d_m^2) - d^2 \right] \xrightarrow[m, n \rightarrow +\infty]{} 0 \quad \checkmark$$

3. H complete  $\Rightarrow \exists y_0 \in H$  s.t.  $y_n \rightarrow y_0$

4. K closed  $\Rightarrow y_0 \in K$

5. By the continuity of  $\|\cdot\|$ ,  $\|x_0 - y_0\| = \lim_{n \rightarrow +\infty} \|x_0 - y_n\| = d$

$$\text{so } \|x_0 - y_0\| = \inf_{y \in K} \|x_0 - y\| = \min_{y \in K} \|x_0 - y\|$$

6.  $y_0$  is unique by (PL):

If  $\tilde{y}_0$  another minimum, then with  $x = x_0 - y_0$ ,  $y = x_0 - \tilde{y}_0$

$$\underbrace{\|2x_0 - (y_0 + \tilde{y}_0)\|^2 + \|y_0 - \tilde{y}_0\|^2}_{4\|x_0 - \frac{y_0 + \tilde{y}_0}{2}\|^2} = 2d^2 + 2d^2$$

$$4\|x_0 - \frac{y_0 + \tilde{y}_0}{2}\|^2 \geq 4d^2$$

$$\Rightarrow \|y_0 - \tilde{y}_0\|^2 \leq 0 \Rightarrow y_0 = \tilde{y}_0$$



### 3.2 Orthogonality and the Projection Theorem

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Recall: Last time we

(1) defined orthogonality  $x \perp y \Leftrightarrow (x, y) = 0$  for  $x, y \in H$

(2) discussed  $P_K : H \rightarrow K$  projection onto  $K$  closed, convex

Thm2: (Projection theorem) Let  $X \subset H$  be a closed subspace ( $x \notin X$ )

in Hilbert. Then  $H = X \oplus X^\perp$ ; that is, every

element  $x \in H$  can be uniquely decomposed as

$$x = y + z \text{ with } y \in X, z \in X^\perp$$

where we denote  $y = P_X x$ ,  $z = P_{X^\perp} x$  projections onto  $X, X^\perp$

N.B.  $X^\perp$  is called the orthogonal complement of  $X$

and  $X, X^\perp$  are said to be complementary subspaces

Proof:

1.  $X$  is a closed convex set, so  $y = P_X x \in X$

is well defined by Thm1 ( $y$  minimizes  $\text{dist}(x, X)$ )

2. Set  $z = x - y$  so that  $x = y + z$ . We need  $z \in X^\perp$

3. The minimizing property of  $y$  means:

•  $\forall v \in X, \forall \lambda \in \mathbb{C}$

$$\|z\|^2 = \|x - y\|^2 \leq \|x - (y + \lambda v)\|^2 \stackrel{\substack{\text{defn } z \\ \text{defn } z}}{=} \|z - \lambda v\|^2$$

$$\Rightarrow \|z\|^2 \leq \|z\|^2 - 2\operatorname{Re}(z, \lambda v) + |\lambda|^2 \|v\|^2$$

$$\Rightarrow |\lambda|^2 \|v\|^2 - 2\operatorname{Re}(z, \lambda v) \geq 0 \quad (*)$$

• Pick  $\lambda = \varepsilon(z, v)$  with  $\varepsilon > 0$  so that

$$\begin{cases} \operatorname{Re}(z, \lambda v) = \operatorname{Re}[\overline{\varepsilon(z, v)}(z, v)] = \varepsilon |(z, v)|^2 \\ |\lambda|^2 = \varepsilon^2 |(z, v)|^2 \end{cases}$$

$$\Rightarrow \varepsilon^2 |(z, v)|^2 \|v\|^2 - 2\varepsilon |(z, v)|^2 \geq 0$$

$$\varepsilon |(z, v)|^2 (\underbrace{\varepsilon \|v\|^2 - 2}_{\text{if } \varepsilon \text{ small}}) \geq 0$$

$$\Rightarrow (z, v) = 0 \quad \forall v \in X \quad \Rightarrow z \in X^\perp$$

4. The decomposition is unique since, if  $x = y + z = y' + z'$  with  $y' \in X$ ,  $z' \in X^\perp$ , then

$$y - y' = z - z' \in X \cap X^\perp = \{0\}$$

[each  $v \in X \cap X^\perp$  satisfies  $(v, w) = 0 \quad \forall w \in X$

and with  $w = v$  :  $(v, v) = 0 \Rightarrow v = 0$ ]



N.B. Since  $X^\perp$  is closed, convex one could start from  $z = P_{X^\perp}x$ .

Remark! Since  $X \perp X^\perp$  one also has

$$\begin{aligned} \|x\|^2 &= \|y\|^2 + \|z\|^2 \text{ in the decomposition of Thm 2} \\ &= \|P_X x\|^2 + \|P_{X^\perp} x\|^2 \end{aligned}$$

Cor 1: If  $X \subsetneq H$  a proper closed subspace, there exists a unit normal vector  $v$  to  $X$ ; that is,

$v \in H$  s.t.  $\|v\|=1$  and  $v \in X^\perp$

Proof: pick any  $x \notin X$  and take  $z = P_{X^\perp}x \in X^\perp$

notice that  $z \neq 0$  and take  $v = \frac{z}{\|z\|}$



Example:  $H = L^2(E)$  with  $E \in M(\mathbb{R}^n)$ ,  $0 < |E| < +\infty$

$X = \{c: E \rightarrow \mathbb{R} \text{ constant function}\}$

$$X^\perp = \{f \in L^2(E): \frac{1}{|E|} \int_E f dx = 0\}$$

$$f \in X^\perp \Leftrightarrow \int_E c f dx = 0 \quad \forall c \text{ constant}$$

$$\Leftrightarrow \text{cl}(E) \frac{1}{|E|} \int_E f dx = 0$$

### 3.3. Duality in Hilbert Spaces

- Goals:
1. Representation formula for the topological dual of a Hilbert space
  2. Applications

Def 1: Let  $H$  be a Hilbert space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ .

$H' = \{ l: H \rightarrow \mathbb{F} \text{ linear, bounded} \}$  where boundedness means:  $\exists C > 0$  s.t.

$$|l(x)| \leq C \|x\| \quad \forall x \in H$$

Remark 1: Since  $H$  is a normed vector space we get

(a)  $l$  linear, bounded  $\Rightarrow$   $l$  continuous

$$[x_j \rightarrow x \text{ in } H \Rightarrow l(x_j) \rightarrow l(x) \text{ in } \mathbb{F}]$$

(b)  $H'$  is a normed vector space with

$$\|l\|_{H'} = \sup_{\substack{x \in H \\ x \neq 0}} \frac{|l(x)|}{\|x\|} = \sup_{\|x\|=1} |l(x)|$$

and  $H'$  is complete

Remark 2: In Hilbert spaces, there are many "built-in" bounded linear functionals. In particular:

$$\forall y \in H \text{ fixed: } l_y(x) := (x, y)$$

•  $l_y$  is linear in  $x$

•  $|l_y(x)| = |(x, y)| \leq \|x\| \|y\|$  is bounded.

Thm 1 (Riesz Representation Theorem) Let  $H$  be Hilbert.

$\forall l \in H' \exists! y \in H$  such that  $l = l_y$ ; that is,

$$l(\cdot) = (\cdot, y) \quad \text{for a unique } y \in H$$

Proof: 1. If  $\ell \geq 0$ , then  $y = 0$  will do

2. If  $\ell \neq 0$ , consider the Kernel of  $\ell$

$$N \stackrel{\text{def}}{=} \{x \in H : \ell(x) = 0\}$$

•  $N$  is a closed subspace ( $\ell$  linear, continuous)

•  $N \subsetneq H$  since  $\ell \neq 0$

• By Cor 3.2.1  $\exists v \in N^\perp$  with  $\|v\|=1$  ( $\ell(v) \neq 0$ )

$$N \cap N^\perp = \{0\}$$

Claim:  $\forall x \in H$  one has  $x - \frac{\ell(x)}{\ell(v)} v \in N$

$$v \neq 0$$

$$\text{in fact, } \ell(x) - \frac{\ell(x)}{\ell(v)} \ell(v) = 0$$

• Hence  $(x - \frac{\ell(x)}{\ell(v)} v, v) = 0 \quad \forall x \in H$

$$\boxed{v \in N^\perp}$$

$$\Rightarrow (x, v) - \frac{\ell(x)}{\ell(v)} (v, v) = 0 \quad \text{but } (v, v) = 1$$

$$\Rightarrow (x, v) - \frac{\ell(x)}{\ell(v)} = 0$$

$$\Rightarrow (x, v) \underbrace{(v)}_{\ell(v)} = \ell(x)$$

$$(x, \overline{\ell(v)} v)$$

$$\text{so } \ell(x) = (\cdot, \overline{\ell(v)} v) \quad \text{if } y = \overline{\ell(v)} v$$

3.  $y$  is unique? if  $\ell(x) = (x, y) = (x, \tilde{y}) \quad \forall x \in H$

$$(x, y - \tilde{y}) = 0 \quad \forall x \in H$$

$$\Rightarrow y - \tilde{y} = 0 \in H$$

□

Remark: For  $H = L^2(E, \mathbb{R})$ , we have seen this result already but with a much more involved proof. We said more there:  $H \cong H'$  which is also true here.

Corollary: If  $H$  is a Hilbert space, then

(a) The map  $\sigma: H \rightarrow H'$  is (anti)-linear, bounded, bijective  
 $y \mapsto \ell_y$

with inverse (the Riesz map)

$$R = \sigma^{-1}: H' \rightarrow H  
\ell \mapsto y \text{ s.t. } \ell = \ell_y$$

which is (anti)-linear, bounded.

(b)  $H'$  is a Hilbert space w.r.t. the inner product

$$(\ell_1, \ell_2)_{H'} = (R\ell_1, R\ell_2)_H = (y_1, y_2)_H$$

and  $\sigma, \sigma^{-1} = R$  are isometric isomorphisms

- proof: (a)
- $\sigma(\lambda y) = \ell_{\lambda y} = \bar{\lambda} \ell_y \quad (\cdot, \lambda y) = \bar{\lambda} (\cdot, y)$
  - $\sigma(y_1 + y_2) = \ell_{y_1+y_2} = \ell_{y_1} + \ell_{y_2} \quad (\cdot, y_1 + y_2) = (\cdot, y_1) + (\cdot, y_2)$
  - $\|\ell_y\|_{H'} = \sup_{\|x\|=1} |\ell_y(x)| = \sup_{\|x\|=1} |(x, y)| = \|y\|_H$
  - $\ell_y = \ell_{y'} \Leftrightarrow (x, y) = (x, y') \quad \forall x \in H$   
 $\Rightarrow (x, y - y') = 0 \quad \forall x$   
 $\Rightarrow y - y' = 0 \quad (\sigma \text{ injective})$
  - $\sigma$  surjective by Thm 1 (RRT)
  - $R$  anti-linear, bounded (Exercise)

(b) We know  $H'$  is Banach (dual of a normed V. S.)

Exercise:  $\{(\cdot, \cdot)\}_{H'}$  is an inner product

$$\|\sigma y\|_{H'} = \|y\|_H \text{ and } (\sigma y, \sigma z)_{H'} = (y, z)_H$$

□

N.B.  $R = \sigma^{-1}: H' \rightarrow H$  is called the canonical isomorphism between  $H'$  and  $H$ . There are others and one important class comes from considering

$$B: H \times H \rightarrow \mathbb{F} \quad (\mathbb{F} = \mathbb{R}, \mathbb{C})$$

sesquilinear / bilinear.

Thm 2: (Lax-Milgram) Let  $B: H \times H \rightarrow \mathbb{F}$  (F) satisfy

- (i)  $B$  is sesquilinear (bilinear)
- (ii)  $B$  is bounded:  $\exists \alpha > 0 \quad |B(x, y)| \leq \alpha \|x\| \|y\|$
- (iii)  $B$  is coercive:  $\exists \beta > 0 \quad |B(x, x)| \geq \beta \|x\|^2$

Then  $\forall l \in \mathbb{F} \exists! y \in H$  s.t.

$$(LM) \quad l(x) = B(x, y) \quad \forall x \in H$$

Remark: when  $B(x, y) = (x, y)_H$  this is exactly the RFT

as  $(\cdot, \cdot)_H = B(\cdot, \cdot)$  satisfies (i), (ii), (iii) with  $\alpha = \beta = 1$

Proof: 1.  $\forall y \in H$  fixed  $B(\cdot, y): H \rightarrow \mathbb{F}$

$$x \mapsto B(x, y)$$

is a bounded linear functional on  $H$  by (ii) and (iii)

$\Rightarrow \forall y \in H \exists! z \in H$  s.t.  
RFT

$$(*) \quad B(x, y) = (x, z) \quad \forall x \in H$$

Define  $T: H \rightarrow H$  by  $T(y) = z$  so that

$$\boxed{(\forall x) \quad B(x, y) = (x, T(y)) \quad \forall x \in H}$$

2.  $T$  is linear: so  $T(H)$  is a vector subspace

$$\bullet \quad \underline{T(\lambda y) = \lambda T(y)}? \quad \underline{\forall x: B(x, \lambda y) = (x, T(\lambda y))} \text{ by (*)}$$

$$\begin{aligned} \lambda B(x, y) &= \lambda (x, T(y)) \\ &= (x, \lambda T(y)) \end{aligned}$$

$$\bullet \quad \underline{T(y_1 + y_2) = T(y_1) + T(y_2)}? \quad \underline{\forall x \in H}$$

$$\begin{aligned} B(x, y_1 + y_2) &= (x, T(y_1 + y_2)) \quad \text{by (*)} \\ &\quad " \end{aligned}$$

$$\begin{aligned} B(x, y_1) + B(x, y_2) &= (x, T(y_1)) + (x, T(y_2)) \\ &= (x, T(y_1) + T(y_2)) \end{aligned}$$

3.  $T$  is injective:  $\forall y \in H$

$$\beta \|y\|^2 \leq |B(y, y)| = |(y, T(y))| \leq \|y\| \|T(y)\| \quad (\text{CSJ})$$

$$\Rightarrow \boxed{(1) \quad \beta \|y\| \leq \|T(y)\|} \quad \begin{array}{l} \text{divide by } \|y\| \text{ if } y \neq 0 \\ (1) \text{ holds if } y=0 \quad (T(0)=0) \end{array}$$

$$\Rightarrow T \text{ injective} \quad (T(y)=0 \Rightarrow y=0)$$

Linear

4.  $T$  is bounded (and hence continuous)

$$\forall y \in H \quad \|T(y)\|^2 = (T(y), T(y)) = B(T(y), y) \quad (\text{CSJ})$$

$$\leq \alpha \|T(y)\| \|y\| \quad (\text{ii})$$

$$\Rightarrow \boxed{(2) \quad \|T(y)\| \leq \alpha \|y\|}$$

again if  $T(y)=0$

then  $y=0$  and (2) is ok.

5.  $T(H) \subset H$  is a closed subspace:

- Let  $z_n = T(y_n) \rightarrow w \in H$ . Then:

$$\boxed{(3) \quad B(x, y_n) = (x, T(y_n)) \rightarrow (x, w) \quad \forall x \in H}$$

- By (1)

$$\beta \|y_n - y_m\| \leq \|T(y_n - y_m)\| = \|z_n - z_m\| \rightarrow 0$$

Since  $\{z_n\}_{n \in \mathbb{N}}$  convergent in  $H$  complete is Cauchy

Hence  $\{y_n\}_{n \in \mathbb{N}}$  is Cauchy in  $H$

$\Rightarrow \exists y \in H$  s.t.  $y_n \rightarrow y$  in  $H$

$$\overline{(3)} \quad \left\{ \begin{array}{ll} B(x, y_n) \rightarrow (x, w) & \forall x \in H \\ B(x, y_n) \rightarrow B(x, y) & \forall x \in H \end{array} \right. \quad \begin{array}{l} B(x, \cdot) \text{ is continuous} \\ \parallel \end{array}$$

$$\Rightarrow \boxed{w = T(y)} \quad (x, T(y))$$

6  $T(H) = H$  ( $T$  surjective)

If  $T(H) \neq H$  then  $\exists v \in T(H)^\perp$  with  $\|v\| \neq 0$  (Proj.Thm)

$$\Rightarrow \beta \|v\|^2 \leq |B(v, v)| = |\underbrace{\langle v, \underbrace{T(v)}_{T(H)^\perp} \rangle|}_{T(H)^\perp} = 0 \quad \underline{\text{absurd}}$$

coercivity

7. Apply RRT to  $\ell$   $\exists! z \in H$  such that

$$\ell(x) = (x, z) \quad \forall x \in H$$

But for each  $z$ ,  $\exists! y$  st.  $T(y) = z$  [  $T: H \rightarrow H$  isomorphism ]

$$\text{so } \ell(x) = (x, T(y)) = B(x, y) \quad \forall x \in H$$

Remark: The map  $H' \rightarrow H$  st  $\ell(\cdot) = B(\cdot, y)$   
 $\ell \mapsto y$

is (another) isomorphism of  $H'$  and  $H$  where

$$\beta \|y\| \leq \|T(y)\| = \|z\| = \|\ell\|_{H'}$$

$\uparrow$  Riesz vector for  $\ell$

$$\text{so } \|y\| \leq \frac{1}{\beta} \|\ell\|_{H'}$$

Remark: The Lax-Milgram Thm is a key tool for solving  
equations in Hilbert spaces. (weak formulations of PDE problems)

N.B. Gave also  $\Delta u = f$  example