

3.4 Orthonormal Systems

EE 18

36/11/17

Point: Examine the notions of linear independence, spanning and bases in infinite dimensional Hilbert spaces.

Def1: Given any subset S of a vector space X over \mathbb{F} the (linear) span of S is

$$\text{span}(S) = \{x = \sum_{i=1}^n c_i x_i : c_i \in \mathbb{F}, x_i \in S, n \in \mathbb{N}\}$$

i.e. the set of all finite linear combinations of elements in S

N.B. If $S = \{x_1, \dots, x_N\}$ is finite, denote

$$\text{span}(S) = \langle x_1, \dots, x_N \rangle$$

Remark1: $\text{span}(S)$ is also:

... (a) $\bigcap_{x \in A} Y_x$ with Y_x linear subspace of X , $S \subseteq Y_x \forall x$.

... (b) the smallest linear subspace containing S

Def2: Let $S = \{x_a\}_{a \in A} \subset H$ with H Hilbert over \mathbb{F}

(a) S is an orthogonal system if $(x_a, x_b) = 0 \quad \forall a \neq b$

(b) S is an orthonormal system if $(x_a, x_b) = \delta_{ab}$; that is

$$x_a \perp x_b \quad \forall a \neq b \quad \text{and} \quad \|x_a\| = 1 \quad \forall a \in A$$

Def3: An o.n. system S is

(a) complete if $\overline{\text{span}(S)}^H = H$ (closure in $\|\cdot\|$)

(b) maximal if $y \perp S \Rightarrow y = 0$

(i.e. $\nexists y \neq 0$ in H with $y \perp S$)

Prop1: Let $S = \{x_a\}_{a \in A}$ be an o.n. system in H

(a) $\sum_{i=1}^N c_i x_{a_i} = 0 \Rightarrow c_i = 0 \quad \forall i = 1, \dots, N$ (linear independence)

(b) S complete $\Leftrightarrow S$ maximal

Pt. exercise

Remark 2 From this proposition, if S is an o.n. system in H

$$(a) \dim H = n \text{ (finite)} \Rightarrow \begin{cases} \text{card}(S) \leq n \\ \text{card}(S) = n \text{ if } S \text{ is complete (maximal)} \end{cases}$$

(b) If $\dim H = \infty$, there is no a priori bound on $\text{card}(S)$

Ex: Let A be an arbitrary set and consider

$$l^2(A) = \{x = \{x_a\}_{a \in A} : x_a \in \mathbb{F}, x_a \neq 0 \text{ for almost countably many } a \in A, \sum_{a \in A} |x_a|^2 < +\infty\}$$

$$\langle x, y \rangle = \sum_{a \in A} x_a \bar{y}_a$$

$$S = \{e^{(b)}\}_{b \in A} \quad e_a^{(b)} = \delta_{ab} \quad \text{just 1 nonzero element}$$

- S is an o.n. system: $\langle e^{(b)}, e^{(c)} \rangle = \sum_{a \in A} \delta_{ba} \delta_{ca} = \delta_{bc}$

- S is maximal (hence complete):

$$y \perp e^{(b)} \quad \forall b \in A \Rightarrow 0 = \langle y, e^{(b)} \rangle = y_b \quad \forall b \in A$$

$$\Rightarrow y = 0$$

Def 4: A Hilbert space H is separable if there exists a countable and complete subset $S \subset H$; that is,

$$S = \{x_n\}_{n \in \mathbb{N}} : \overline{\text{span}(S)}^H = H$$

so that: $\forall x \in H, \forall \varepsilon > 0 \exists N = N(x, \varepsilon), \{c_n\}_{n=1}^N \subset \mathbb{F}$ st.

$$\|x - \sum_{n=1}^N c_n x_n\| < \varepsilon$$

Thm 1: Let H be an infinite dimensional separable Hilbert space. Then:

(a) Every o.n. system S is at most countable.

(b) There exists a complete o.n. system.

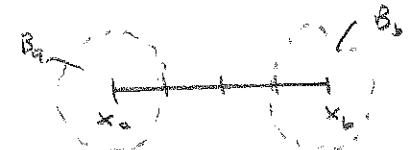
Proof:

(a) By contradiction, suppose $S = \{x_\alpha\}_{\alpha \in A}$ is an uncountable o.n. system

• $\forall \alpha \in A$ consider $B_\alpha = B_{\frac{1}{2}}(x_\alpha)$ and one has

$$B_\alpha \cap B_\beta = \emptyset \quad \forall \alpha \neq \beta$$

in fact $\|x_\alpha - x_\beta\|^2 = \|x_\alpha\|^2 + \|x_\beta\|^2 - 2(x_\alpha, x_\beta) = 2$



• H separable $\Rightarrow \exists \{y_n\}_{n \in \mathbb{N}}$ countable dense

$$\Rightarrow \exists \alpha^* \in A : y_n \notin B_{\alpha^*} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \|x_{\alpha^*} - y_n\| \geq \frac{1}{2} \quad \forall n \in \mathbb{N} \quad \text{to density}$$

(b) Consider $S = \{x_n\}_{n \in \mathbb{N}}$ complete, countable (H separable)

1. Inductively, eliminate from S all elements x_n s.t.

$$x_n \in \langle x_1, \dots, x_{n-1} \rangle$$

and call the new collection $S_I = \{z_n\}_{n \in \mathbb{N}}$

N.B. S_I is infinite since $\dim H = \infty$

2. Use the Gram-Schmidt process

• pick $e_1 = \frac{1}{\|z_1\|} z_1$ so that $\|e_1\|=1$ and

$$H_1 = \langle e_1 \rangle = \langle z_1 \rangle$$

• Call $H_2 = \langle e_1, z_2 \rangle$ where $z_2 \neq c e_1 \quad c \in \mathbb{R}$

project z_2 onto H_1 to find $u_2 = (z_2, e_1)e_1$

and consider $e_2 = \frac{1}{\|z_2 - u_2\|} (z_2 - u_2)$ with $\begin{cases} \|e_2\|=1 \\ \langle e_1, e_2 \rangle = H_2 \end{cases}$

• By induction, from $H_n = \langle e_1, \dots, e_n \rangle = \langle z_1, \dots, z_n \rangle$

$$e_{n+1} = \frac{1}{\|z_{n+1} - u_{n+1}\|} (z_{n+1} - u_{n+1}) \quad \text{with } u_{n+1} = P_{H_n}(z_{n+1}) \\ = \sum_{j=1}^n (z_{n+1}, e_j) e_j$$

We never stop after a finite # of steps,
and we get $S_{on} = \{e_n\}_{n \in \mathbb{N}}$ o.n. system w/ $\text{span}(S_{on}) = \text{span}(H)$

Def 5: An orthonormal basis of H is a complete (maximal) o.n. system

N.B Thm 1 says separable Hilbert spaces have an o.n.-basis
Also nonseparable H have o.n.-bases, but instead of the concrete G-S process, one appeals to Zorn's Lemma and the axiom of choice

N.B "Most" useful Hilbert spaces are separable.

Now: Let's consider some additional facts about o.n. systems in H Hilbert related to series expansions

Lemma 1: Let H be Hilbert (not necessarily separable) and $S = \{x_n\}_{n \in \mathbb{N}}$ an o.n.-system (not necessarily maximal)

(a) $\{\langle x, x_n \rangle\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{R})$ with

$$\sum_{n \in \mathbb{N}} |\langle x, x_n \rangle|^2 \leq \|x\|^2 \quad (\text{Bessel's inequality})$$

(b) $\forall \{c_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ and $\forall N \geq 1$

$$\|x - \sum_{n=1}^N c_n x_n\| \geq \|x - \sum_{n=1}^N \langle x, x_n \rangle x_n\|$$

N.B • Best least squares approximation

• (b) $\Rightarrow P_{\langle x_1, \dots, x_N \rangle} x = \sum_{n=1}^N \langle x, x_n \rangle x_n$, generalizes

what we did with $x_n \leftrightarrow e_n$ in GS with $x = z_{n+1}$

proof: ($\mathbb{F} = \mathbb{C}$ case)

1. $\forall c, z \in \mathbb{C}$:

$$(*) |c - z|^2 = (c - z)(\bar{c} - \bar{z}) = |c|^2 + |\bar{z}|^2 - c\bar{z} - \bar{c}z$$

$$\begin{aligned} \text{so } \|x - \sum_{n=1}^N c_n x_n\|^2 &= (x - \sum_{n=1}^N c_n x_n, x - \sum_{k=1}^N c_k x_k) \quad \text{c_n TBA} \\ &= \|x\|^2 + \sum_{n, k=1}^N (c_n x_n, c_k x_k) - \sum_{n=1}^N (c_n x_n, x) \\ &\quad - \sum_{k=1}^N (x, c_k x_k) \end{aligned}$$

but $\langle x_n, x_k \rangle = s_{nk}$ and $c_n \bar{c}_n = |c_n|^2$ so

$$\|x - \sum_{n=1}^N c_n x_n\|^2 = \|x\|^2 + \sum_{n=1}^N [c_n^2 - c_n \langle x, x_n \rangle - \bar{c}_n \langle x, x_n \rangle]$$

and using (*) N times with $c = c_n$, $z = (x, x_n)$

$$(**) \|x - \sum_{n=1}^N c_n x_n\|^2 = \|x\|^2 + \sum_{n=1}^N [c_n - \langle x, x_n \rangle]^2 = \|x\|^2 + \sum_{n=1}^N |\langle x, x_n \rangle|^2$$

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Pick $c_n = \langle x, x_n \rangle$ to yield

$$0 \leq \|x - \sum_{n=1}^N c_n x_n\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n \rangle|^2$$

$$\Rightarrow \sum_{n=1}^N |\langle x, x_n \rangle|^2 \leq \|x\|^2 \quad \forall N \in \mathbb{N}$$

and take $N \rightarrow \infty$ to find (a).

2. In (**) above, with $\{c_n\}_{n \in \mathbb{N}}$ left free

$$\begin{aligned} \|x - \sum_{n=1}^N c_n x_n\|^2 &\geq \|x\|^2 - \sum_{n=1}^N |\langle x, x_n \rangle|^2 \\ &= \|x\|^2 - \left\| \sum_{n=1}^N \langle x, x_n \rangle x_n \right\|^2 \quad \{x_n\} \text{ an o.n. system} \\ &= \|x - \sum_{n=1}^N \langle x, x_n \rangle x_n\|^2 \quad (\text{easy calculations}) \end{aligned}$$

□

Thm 2: Let H be Hilbert over \mathbb{F} , $S = \{x_n\}_{n \in \mathbb{N}}$ an o.n. system and $\{c_n\}_{n \in \mathbb{N}} \subset \mathbb{F}$. Then

$$(a) \sum_{n \in \mathbb{N}} c_n x_n \text{ converges in } H \Leftrightarrow \{c_n\}_{n \in \mathbb{N}} \in l^2(\mathbb{F})$$

$$(b) \left\| \sum_{n \in \mathbb{N}} c_n x_n \right\|^2 = \sum_{n \in \mathbb{N}} |c_n|^2 \quad (\text{Parseval's identity})$$

(c) The convergence in (a) is unconditional

Proof (a) If $x = \sum_{n \in \mathbb{N}} c_n x_n$ in H , then $c_n = (x, x_n)$

• By Lemma 1 [best least squares]

$$\|x - \sum_{n=1}^N (x, x_n) x_n\| \leq \|x - \sum_{n=1}^N c_n x_n\| \rightarrow 0 \quad N \rightarrow \infty$$

$$\text{So } x = \lim_{N \rightarrow \infty} \sum_{n=1}^N (x, x_n) x_n = \sum_{n \in \mathbb{N}} (x, x_n) x_n = \sum_{n \in \mathbb{N}} c_n x_n$$

$$\begin{aligned} \text{• So } \forall k \in \mathbb{N} \quad & \left(\underbrace{\sum_{n \in \mathbb{N}} (x, x_n) x_n}_{\lim_{N \rightarrow \infty} T_N}, x_k \right) = \left(\underbrace{\sum_{n \in \mathbb{N}} c_n x_n}_{\lim_{N \rightarrow \infty} S_N}, x_k \right) \\ & (x, x_k) = c_k \end{aligned}$$

$$2. \text{ Hence } \sum_{n \in \mathbb{N}} |c_n|^2 = \sum_{n \in \mathbb{N}} |(x, x_n)|^2 \leq \|x\|^2 \quad (\text{Bessel's inequality})$$

Lemma 1(e)

$$\Rightarrow \{c_n\}_{n \in \mathbb{N}} \in l^2(\mathbb{F})$$

3. Conversely, if $\{c_n\}_{n \in \mathbb{N}} \in l^2(\mathbb{F})$ then

$$(*) \quad \left\| \sum_{n=M}^N c_n x_n \right\|^2 = \sum_{n=M}^N |c_n|^2 \rightarrow 0 \quad \text{as } M, N \rightarrow \infty$$

(Cauchy criterion for series)

$\Rightarrow \{S_N = \sum_{n=1}^N c_n x_n\}_{N \in \mathbb{N}}$ is Cauchy in H complete

$$\Rightarrow \sum_{n=1}^{+\infty} c_n x_n \text{ converges in } H$$

(b) From (*) with $M = 1$

$$\left\| \sum_{n=1}^N c_n x_n \right\|^2 = \sum_{n=1}^N |c_n|^2 \quad \text{and take } N \rightarrow +\infty$$

(c) See [V] dispense



Thm 3: Let $S = \{x_n\}_{n \in \mathbb{N}}$ be an o.n. system in H Hilbert.

TFAE (a) S is complete ; Prop 1

(b) S is maximal

(c) $\forall x \in H \quad \sum_{n=1}^{\infty} (x, x_n) x_n \rightarrow x \text{ in } H$

(d) $\forall x \in H \quad \sum_{n=1}^{\infty} |(x, x_n)|^2 = \|x\|^2$

3.5 Linear operators on Hilbert spaces

LEZ19

5/12/17

Motivation: We have discussed the notion of complete orthonormal systems (bases) and proved the existence of an O.N. basis for H separable. A general construction of useful bases comes from eigen function bases associated to some linear operator $T: H \rightarrow H$. It will turn out that this is always possible if T is bounded, compact, self-adjoint.

N.B. In esercitazioni, concrete examples of Fourier series will be considered. These are often derived from the ideas we will discuss after.

3.5.1 Bounded linear operators

Def1: A linear operator $T: H_1 \rightarrow H_2$ between Hilbert spaces is called bounded if $\exists C > 0$ s.t.

$$\|Tu\|_{H_2} \leq C \|u\|_{H_1}, \quad \forall u \in H_1,$$

and we will write $T \in \mathcal{L}(H_1, H_2)$. If $H_1 = H_2 = H$, T will be called a linear operator on H and we denote $T \in \mathcal{L}(H)$.

Exercise1: Given $T: H_1 \rightarrow H_2$ linear, TFAE

(a) T is bounded

(b) T is continuous: $u_n \rightarrow u$ in $H_1 \Rightarrow Tu_n \rightarrow Tu$ in H_2

(c) T is continuous in 0: $u_n \rightarrow 0$ in $H_1 \Rightarrow Tu_n \rightarrow 0$ in H_2

Def2: If $T \in \mathcal{L}(H_1, H_2)$, then the (operator) norm of T is

$$\|T\|_{op} = \sup_{\substack{u \neq 0 \\ u \in H_1}} \frac{\|Tu\|_{H_2}}{\|u\|_{H_1}} = \sup_{\substack{u \in H_1 \\ \|u\|_{H_1}=1}} \|Tu\|_{H_2}$$

Exercise2: $(\mathcal{L}(H_1, H_2), \|\cdot\|_{op})$ is a Banach space.

Remark 1: A bounded linear operator is uniquely determined by its values on a dense subspace of H_1 .

Exercise 3: Given $T: D_T \subset H_1 \rightarrow H_2$ with $D_T \subset H_1$ dense subspace and T bounded on D_T . Then $\exists! \tilde{T} \in \mathcal{L}(H_1, H_2)$ s.t.

$$\tilde{T}|_{D_T} = T$$

Remark 2: Exercises 1-3, Definitions 1-2 hold also if $H_1 \hookrightarrow X$ normed v.s. $H_2 \hookrightarrow Y$ Banach space.

N.B An important class of $\mathcal{L}(H)$ are Hilbert-Schmidt operators

Prop 1: Let $K \in L^2(E \times E)$ with $E \in m(\mathbb{R}^n)$ and $H = L^2(E)$ the operator $T: H \rightarrow H$ defined by

$$(*) \quad T(u)(x) = \int_E k(x,y) u(y) dy$$

satisfies:

$$(a) \quad T \in \mathcal{L}(H)$$

$$(b) \quad \|T\|_{op} \leq \|K\|_{L^2(E \times E)}$$

Proof: Notice that $(*)$ does not depend on the representative u of an L^2 class.

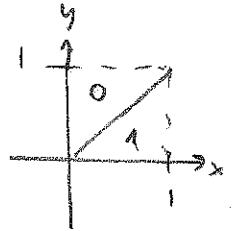
$$\bullet \quad \left| \int_E k(x,y) u(y) dy \right| \stackrel{\text{Holder}}{\leq} \left(\int_E |k(x,y)|^2 dy \right)^{\frac{1}{2}} \|u\|_{L^2(E)}$$

$$\begin{aligned} \bullet \quad \|T\|_{L^2(E)}^2 &= \int_E \left| \int_E k(x,y) u(y) dy \right|^2 dx \\ &\leq \|u\|_{L^2(E)}^2 \int_E \left(\int_E |k(x,y)|^2 dy \right) dx \\ &= \|u\|_{L^2(E)}^2 \|K\|_{L^2(E \times E)}^2 \end{aligned}$$

from which (a) and (b) follow.

(2)

Example: $E = (0, 1)$ $K(x,y) = \begin{cases} 1 & 0 < y < x \\ 0 & \text{else} \end{cases}$



$$\bullet T_u(x) = \int_0^1 K(x,y) u(y) dy = \int_0^x u(y) dy = U(x)$$

where $u \in L^2(E) \Rightarrow u \in L^1(E) \Rightarrow U \in AC(E)$

$$\bullet K \in L^2(E \times E) \text{ with } \|K\|_{L^2(E \times E)}^2 = \int_0^1 \int_0^x dy dx = \int_0^1 x dx = \frac{1}{2}$$

so $T \in X(L^2(E))$ and $\|T\|_{op} \leq \frac{1}{2}$

3.6.2 Adjoint operators

Def 3: Given $T \in X(X, Y)$ between normed vector spaces, the transpose of T is the operator $T^t \in X(Y', X')$ defined by duality:

$$\forall l \in Y': \quad T^t l(u) \doteq l(Tu) \quad \forall u \in X \quad \text{defines } T^t l \in X'$$

$\forall y \in Y', T^t l \in X'$: i.e. (bdd linear functional on X)

$$(a) T^t l(\alpha u + \beta v) \doteq l(T(\alpha u + \beta v)) = l(\alpha Tu + \beta Tv)$$

$$\begin{aligned} &= \alpha l(Tu) + \beta l(Tv) \doteq \alpha T^t l(u) + \beta T^t l(v) \\ &\text{l linear} \end{aligned}$$

$$(b) |T^t l(u)| \doteq |l(Tu)| \leq \|l\|_{Y'}, \|Tu\|_Y \leq \|u\|_X, \|Tu\|_Y \leq \|T\|_{op} \|u\|_X$$

$$\bullet T^t \text{ linear: } T^t(\alpha l_1 + \beta l_2) = \alpha T^t l_1 + \beta T^t l_2 \text{ in } X' \quad (\text{easy exercise})$$

T^t bounded: (b) shows

$$\forall l \in Y', \|T^t l\|_{X'} \leq \|l\|_{Y'} \|T\|_{op}$$

Remark 3: When $X = H_1$ and $Y = H_2$ are Hilbert, we have

the Riesz Representation Theorem:

so have

$$\begin{array}{ccc} H_1 & \xrightarrow{T} & H_2 \\ R_1 \uparrow \downarrow \sigma_1 & \xleftarrow{T^t} & \downarrow \sigma_2 \uparrow R_2 \\ H_1' & \xleftarrow{T^t} & H_2' \end{array}$$

$$\begin{aligned} &H_1 \xrightarrow{\sigma_1} H_1' \\ &u \mapsto l_u = (\cdot, u) \\ &\sigma_1^{-1} = R_1 \text{ Riesz map} \end{aligned}$$

Hence we can define a map $T^* \in \mathcal{X}(H_2, H_1)$ by

$$T^* = R_1 \circ T^t \circ \tau_2$$

Prop2: Given $T \in \mathcal{X}(H_1, H_2)$. Then

- (a) $\exists! T^* \in \mathcal{X}(H_2, H_1)$ s.t. $(Tu, v)_{H_2} = (u, T^*v)_{H_1}$, $u \in H_1, v \in H_2$
- (b) $\|T^*u\|_{op} = \|T\|_{op}$
- (c) $(T^*)^* = T$

proof! (a) If we take $T^* = R_1 \circ T^t \circ \tau_2$, then

$$\begin{aligned} (u, T^*v)_{H_1} &= (u, R_1(T^t(\tau_2 v))_{H_1}) \\ &\stackrel{\text{def}}{=} T^t(\tau_2 v)[u] \quad v \in H_2, \tau_2 v \in H_2', T^t \tau_2 v \in H_1' \\ &= \tau_2 v (Tu) \\ &\stackrel{\text{def } T^t}{=} (Tu, v)_{H_2} \end{aligned}$$

Is T^* unique? Let's say there are 2 such operators T^*, S^*

$$\text{Then } (u, T^*v)_{H_1} = (Tu, v)_{H_2} = (u, S^*v)_{H_1}, \forall u \in H_1, \forall v \in H_2$$

$$\Rightarrow (u, T^*v - S^*v)_{H_1} = 0 \quad \forall u \in H_1 \quad (v \in H_2 \text{ fixed})$$

$$\Rightarrow T^*v - S^*v = 0 \quad \text{in } H_1, \forall v \text{ fixed}$$

(b)/(c) are easy exercises

□

Def4 An operator $T \in \mathcal{X}(H)$ is called self-adjoint

If $T = T^*$; that is,

$$(u, Tv)_{H} = (Tu, v)_{H} \quad \forall u, v \in H$$

Exercise4: A Hilbert-Schmidt operator T is self-adjoint

$$\text{if } K(x, y) = K(y, x) \quad \forall x, y \in E$$

N.B One last ingredient, compactness.

3.5.3 Compact operators in Hilbert spaces

Def 5: $T \in \mathcal{L}(H_1, H_2)$ is called compact if

$\forall \{u_n\}_{n \in \mathbb{N}} \subset H$ bounded, $\{Tu_n\}_{n \in \mathbb{N}} \subset H_2$ has a convergent subsequence.

Exercise 5: T is compact iff

$B \subset H_1$ bounded $\Rightarrow T(B)$ is precompact in H_2
 $(\bar{T(B)} \text{ is compact in } H_2)$

Remark 4: In general, $T \in \mathcal{L}(H_1, H_2)$ is not compact.

Example: $T = \text{Id}$ on H separable

- Take an o.n.b $\{e_n\}_{n \in \mathbb{N}}$ for H . (bounded sequence)
- Suppose that $\{Te_n\}_{n \in \mathbb{N}} = \{e_n\}_{n \in \mathbb{N}}$ has a norm conv. subseq. $\{e_{n_k}\}_{k \in \mathbb{N}}$
- $\|e_{n_k} - e_{n_j}\|_H^2 = (e_{n_k} - e_{n_j}, e_{n_k} - e_{n_j}) = 2 \quad \forall j \neq k$
 So $\{e_{n_k}\}_{k \in \mathbb{N}}$ is not Cauchy in H complete. Absurd.

Remark 5: In finite dimensional Hilbert spaces, bounded linear operators are compact, and a related fact is the following result.

Prop 3: Let $T \in \mathcal{L}(H_1, H_2)$ be given.

(a) T is compact if it has finite rank; i.e.

$T(H_1) \subset V$ with $\dim(V) < +\infty$

(b) T is compact if it is an operator norm limit

of a sequence of finite rank operators; i.e. if

$$\|T_n - T\|_{op} \doteq \sup_{u \in H_1} \|T_n u - Tu\|_{H_2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$\|u\|_{H_1} = 1$

and T_n finite rank

Proof (a) Use Bolzano-Weierstrass on $\nabla = T(H_1)$

$\{Tu_n\}_{n \in \mathbb{N}}$ bdd in $H_1 \Rightarrow \{Tu_n\}_{n \in \mathbb{N}}$ bounded in $\nabla \subset H_2$
 T bdd linear

$\Rightarrow \exists$ convergent subsequence
 $\text{in } \nabla \subset H_2$

(b) Use a Cantor diagonalization argument (exercise) □

Remark If H is separable, then every compact operator T
is a norm limit of finite rank operators

Prop 4: Hilbert-Schmidt operators are compact on $L^2(E)$

Proof: • Take an o.n.b. for $L^2(E)$ separable Hilbert

Then $e_{kj}(x,y) = e_k(x) e_j(y)$ is an o.n.b. for $L^2(E \times E)$

Thus $K(x,y)$ can be written as

$$K(x,y) = \sum_{j,k=1}^{+\infty} c_{kj} e_k(x) e_j(y) \quad \text{with} \quad \sum_{j,k=1}^{+\infty} |c_{jk}|^2 < +\infty$$

• Define

$$\begin{cases} T_N u(x) = \int_E K_N(x,y) u(y) dy \\ K_N(x,y) = \sum_{j,k=1}^N c_{kj} e_k(x) e_j(y) \end{cases}$$

$$\bullet \quad T_N u = \sum_{k=1}^N \tilde{c}_k e_k \quad \tilde{c}_k = \sum_{j=1}^N c_{kj} \int_E g_j(y) dy$$

so has finite rank, and hence compact.

• $\|T_n - T\|_{op} \rightarrow 0$ as $n \rightarrow \infty$ (exercise) □