

### 3.5 Linear operators on Hilbert spaces

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3.5.1 Bounded linear operators  $T: H_1 \rightarrow H_2$  ( $T \in \mathcal{L}(H_1, H_2)$ )

$$\|T\|_{H_1 \rightarrow H_2} < C \|T\|_{H_1}$$

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3.5.2 Adjoint operators  $T \in \mathcal{L}(H_1, H_2)$  has  $T^* \in \mathcal{L}(H_2, H_1)$

$$(Tu, v)_{H_2} = (u, T^*v)_{H_1}$$

3.5.3 Compact operators  $T \in \mathcal{L}(H_1, H_2)$  s.t.

$\{u_n\}_{n \in \mathbb{N}}$  bdd  $\Rightarrow \{Tu_n\}_{n \in \mathbb{N}}$  has  
in  $H_2$  convergent subseq. in  $H_2$

Prop 3: Let  $T \in \mathcal{L}(H_1, H_2)$  be given.  $T$  is compact if

(a)  $T$  has finite rank; that is,  $T(H_1) \subset V$  with  $\dim V < \infty$

(b)  $T = \lim_{j \rightarrow \infty} T_j$  in  $\mathcal{L}(H_1, H_2)$  with  $T_j$  finite rank

Proof: (a) Bolzano-Weierstrass in  $V$  done

$\{u_n\}_{n \in \mathbb{N}}$  bdd in  $H_1 \Rightarrow \{Tu_n\}_{n \in \mathbb{N}} \subset V$  ( $\dim V < \infty$ )  
 $T$  bdd  $\Rightarrow \{Tu_n\}_{n \in \mathbb{N}}$  bdd in  $H_2$

(b) use a Cantor diagonalization argument (exercise)

Remark 6: If  $H$  is separable, every  $T \in \mathcal{L}(H)$  compact  
is a norm limit of finite rank operators.

Prop 4: Hilbert-Schmidt operators are compact on  $L^2(E)$

Proof: Take an o.n.b.  $\{e_j\}_{j \in \mathbb{N}}$  for  $L^2(E)$  separable

then  $e_{j,k}(x, y) = e_j(x) \delta_{jk}(y)$  is an o.n.b. for  $L^2(E \times E)$

so can write kernel  $K(x, y) = \sum_{j,k=1}^n c_{jk} e_j(x) \delta_{jk}(y)$

with  $\sum_{j,k=1}^n |c_{jk}|^2 < \infty$

• Define  $T_N u(x) = \int_E K_N(x, y) u(y) dy$        $K_N = \sum_{j,k=1}^N c_{jk} e_j(x) \delta_{jk}$   
 $= \sum_{j=1}^N \tilde{c}_j e_j(x)$        $\tilde{c}_j = \sum_{k=1}^N c_{jk} \left( \int_E e_j(y) e_k(y) dy \right)$

•  $T_N$  finite rank, so compact and  $\|T_N - T\|_F \rightarrow 0$  as  $N \rightarrow \infty$  (exercise)

### 3.6 The spectral theorem

Goal: Prove that  $T \in \mathcal{L}(H)$  self-adjoint and compact  
on  $H$  separable determines an o.n.b. for  $H$  of eigenfunctions  
of  $T$

Def1: Let  $T \in \mathcal{L}(H)$  be given.  $\lambda \in \mathbb{F}$  is called an  
eigenvalue of  $T$  if there exists  $u \in H$  nonzero s.t.

$$Tu = \lambda u \quad \text{in } H$$

$u$  is called an eigenvector (eigenfunction) of  $T$  and

we denote by  $E_\lambda = \{u \in H : Tu = \lambda u\}$  the eigenspace associated

Thm1: (The spectral theorem) (Let  $T \in \mathcal{L}(H)$  be compact,  
self-adjoint on  $H$  separable. Then

(a) All eigenvalues of  $T$  are real, and eigenfunctions  
corresponding to distinct eigenvalues are orthogonal

(b)  $\dim(E_\lambda) < +\infty \quad \forall \lambda \neq 0$  [finite rank ops have  $E_0$   
 $\infty$  dimensional if  $\dim H = \infty$ ]

(c) If  $\text{ev}(T) = \{\lambda \in \mathbb{R} : \lambda \text{ is an eval of } T\}$  then

$\text{ev}(T)$  is at most countable and  $0$  is the only possible  
accumulation point

(d) There is an o.n.b. for  $H$  made up of eigenfunctions of  $T$

Proof: (a)  $\lambda(u, u) = (\lambda u, u) = (Tu, u) = (u, Tu) = (u, \lambda u)$

S.P.                    eval                    SA                    eval

$= \bar{\lambda}(u, u)$

but  $(u, u) = \|u\|^2 \neq 0 \Rightarrow \lambda = \bar{\lambda}$  so  $\lambda \in \mathbb{R}$

Let  $Tu_k = \lambda_k u_k$ ,  $k=1, 2$  with  $\lambda_1 \neq \lambda_2$

$$\lambda_1(u_1, u_2) = (\lambda_1 u_1, u_2) = (Tu_1, u_2) = (u_1, Tu_2)$$

$$= (u_1, \lambda_2 u_2) = \lambda_2(u_1, u_2) \quad (\lambda_2 \in \mathbb{R})$$

$$\Rightarrow (\lambda_1 - \lambda_2)(u_1, u_2) = 0 \quad \underset{\lambda_1 \neq \lambda_2}{\Rightarrow} \quad \underline{(u_1, u_2) = 0}$$

(b) Suppose not. Then  $\exists S = \{e_n\}_{n \in \mathbb{N}} \subset E_\lambda$  an o.n. system.

$T \in \mathcal{L}(H)$

$\cdot S \subset H$  is bounded in  $H \Rightarrow \{Te_n\}_{n \in \mathbb{N}}$  bounded in  $H$

$\xrightarrow{T \text{ compact}} \exists \{Te_{n_k}\}_{k \in \mathbb{N}}$  subseq converging in  $H$

But  $Te_{n_k} = \lambda e_{n_k}$  with  $\lambda \neq 0$

so  $\{e_{n_k}\}_{k \in \mathbb{N}}$  is a converging sequence

Absurd:  $\|e_{n_k} - e_j\|^2 = 2 \quad \forall k \neq j$

(c) To show  $\text{ev}(T)$  at most countable and  $\text{ev}(T) \subset \{0\}$   
it suffices to show

$$\boxed{\forall \mu > 0 \quad \#\{\lambda \in \text{ev}(T) : |\lambda| > \mu\} < +\infty}$$

$\cdot$  Suppose not. Then  $\exists \{e_n\}_{n \in \mathbb{N}}$  associated to  $\{\lambda_n\}_{n \in \mathbb{N}}$   
with  $\begin{cases} |\lambda_n| > \mu \quad \forall n \in \mathbb{N} \\ \lambda_j \neq \lambda_k \text{ if } j \neq k \end{cases}$

$\therefore \{e_n\}_{n \in \mathbb{N}}$  an o.n. system! (nonzero orthogonal eigenfunctions corresponding to distinct evals)

$\Rightarrow \exists$  convergent subsequence of  $\{Te_n\}_{n \in \mathbb{N}}$

$T$  compact

But  $Te_n = \lambda_n e_n$  and

$$\| \lambda_{n_k} e_{n_k} - \lambda_j e_j \|_H^2 = \lambda_{n_k}^2 + \lambda_j^2 \geq 2\mu^2 > 0$$

absurd Cauchy Criterion

(d) wlog  $T \neq 0$ : (else  $\text{ev}(T) = \{0\}$  and  $E_0 = H$  separable)

Fact 1:  $T \neq 0 \Rightarrow \|T\|_F$  or  $-\|T\|_F$  is an eigenvalue

exercise.

$\cdot$  So  $S = \overline{\text{span}\{e_n\}_{n \in \mathbb{N}}} \neq \emptyset$

$e_n \neq 0$  eigenfunction of  $T$

Claim:  $S = H$

If not,  $H = S \oplus S^\perp$  with  $S, S^\perp \neq \{0\}$

Fact 2:  $TS \subset S$  and  $TS^\perp \subset S^\perp$

- $T\left(\sum_{k=1}^N c_k e_k\right) = \sum_{k=1}^N c_k \lambda_k e_k \in S$

can pass to the limit as  $N \rightarrow +\infty$  so  $TS \subset S$

- Let  $u \in S^\perp$ , then  $\forall v \in S$

$$(Tu, v) = (u, Tv) = 0 \quad \begin{matrix} \text{v.e.s.t} \\ \text{s.t. } u \\ S^\perp \subset S \text{ (above)} \end{matrix} \Rightarrow Tu \in S^\perp$$

- Hence consider  $T_1 = T|_{S^\perp} : S^\perp \rightarrow S^\perp$

$T_1 \in \mathcal{Z}(S^\perp)$  is compact, self-adjoint on  $(S^\perp, (\cdot, \cdot)_H)$   
a separable Hilbert space

- $T_1 \neq 0$ , so it has a non zero eval  $\lambda$  ( $\|T_1\|_{op}, -\|T_1\|_{op}$ )

so,  $T_1$  has an eigenfunction  $T_1 u = \lambda u$

which is then an eigenfunction for  $T$  in  $S^\perp$   
absurd.  $S$  had all the indep eigenfunctions

□

Final Remarks: We could say a lot more

(a) Fredholm alternative  $Tu - fu = f$    
 ↗  $\exists!$  soln if  $\lambda \notin \text{ev}(T)$   
 ↗  $\ker(T - \lambda I)$  finite dim  
 and  $\exists u \Leftrightarrow f \perp \ker(T - \lambda I)^*$

(b) More complete description of spectral theory /  
examples/ applications

N.B Impinging on / Elements of Functional Analysis  
 ↗ PDE

### 3.8. Pointwise convergence of Fourier Series

Point! If  $\{e_n\}_{n \in \mathbb{N}}$  is an o.n.b. for  $H$  separable,  
we have seen  $u = \sum_{n \in \mathbb{N}} c_n e_n$  with  $H$ -norm convergence

Important examples of complete o.n. systems are

$$(1) \quad \{e^{ikx}\}_{k \in \mathbb{Z}} \quad (\text{values in } \mathbb{C})$$

$$(2) \quad \{\cos(nx), \sin(nx)\}_{n \in \mathbb{N}} \quad (\text{values in } \mathbb{R})$$

which generate classical Fourier series.

• examine pointwise convergence for them

Fact!  $\{e^{ikx}\}_{k \in \mathbb{Z}}$  is a complete o.n. system for  $H = L^2(\mathbb{Q}, \frac{dx}{2\pi})$

where  $\mathbb{Q} = [-\pi, \pi] \cong \mathbb{T}^1$  (circle = 1-torus) values in  $\mathbb{C}$

$$\text{and } (f, g) = \frac{1}{2\pi} \int_{\mathbb{Q}} f \bar{g} dx$$

$$\begin{aligned} \langle e^{ikx}, e^{ijx} \rangle &= \frac{1}{2\pi} \int_{\mathbb{Q}} e^{ikx} e^{-ijx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\underbrace{\cos((k-j)x)}_{\text{odd in } x \text{ if } j \neq k} + i \sin((k-j)x)] dx \\ &= \begin{cases} 1 & j \neq k \\ 0 & j = k \end{cases} \end{aligned}$$

• various proofs of completeness [Stone-Weierstrass, summability kernels]

$$\bullet S_N(f) = \sum_{k=-N}^N \hat{f}(k) e^{ikx} \quad \text{where}$$

$$\hat{c}_k = \hat{f}(k) = (f, e^{ikx}) = \frac{1}{2\pi} \int_{\mathbb{Q}} f(x) e^{-ikx} dx$$

$$\|S_N(f) - f\| \rightarrow 0 \text{ as } N \rightarrow +\infty \quad \forall f \in L^2(\mathbb{Q}, \frac{dx}{2\pi})$$

N.B. In the real case  $L^2(\mathbb{Q}, \mathbb{R})$  one writes

$$S_N f(x) = \frac{a_0(t)}{2} + \sum_{n=1}^{+\infty} [a_n(t) \cos(nx) + b_n(t) \sin(nx)]$$

with  $a_0(t) = 2 \hat{f}(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$$\left\{ \begin{array}{l} a_n(t) = \hat{f}(n) + \hat{f}(-n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n \in \mathbb{N} \\ b_n(t) = i[\hat{f}(n) - \hat{f}(-n)] = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \end{array} \right.$$

$$b_n(t) = i[\hat{f}(n) - \hat{f}(-n)] = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Remark 1: Since all terms in  $S_N f(x)$  are  $2\pi$ -periodic and continuous a special role is played by

$$C_{\text{per}}^0(\bar{\mathbb{Q}}, \mathbb{F}) = \{ f: \bar{\mathbb{Q}} \rightarrow \mathbb{F} \text{ continuous with } f(-x) = f(x) \}$$

These functions have a continuous extension to  $2\pi$ -periodic functions on  $\mathbb{R}$ .

Or can be thought of as  $C^0(\pi', \mathbb{F})$

Remark 2: Since  $\{e^{ikx}\}, \{1, \cos(kx), \sin(kx)\}$  are bounded on  $\mathbb{Q}$  we can compute  $\hat{f}(k), a_0(t), a_n(t), b_n(t)$  for  $f \in L^p(\mathbb{Q}, \mathbb{F}) \quad \forall p \geq 1$  (not just  $L^2$ )

Question: Under what conditions can we say

$$S_N f(x) \rightarrow f(x) \quad \text{for a given } x \in \mathbb{Q}?$$

Theorem 1: Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be  $2\pi$ -periodic and piecewise regular

[  $f$  bounded on  $[-\pi, \pi]$ , at most finite # of jump discontinuities in  $x_1, \dots, x_m$  with  $f(x_j^\pm)$  finite,  $f' \in BC(\mathbb{R} \setminus S)$  with  $S = \{x_j\}_{j=1}^m \cup \{y_k\}_{k=1}^n$  ]

Then  $\forall x_0 \in \mathbb{R}$ ,

$$S_N f(x_0) \rightarrow \frac{f(x_0^+) + f(x_0^-)}{2} \quad \text{as } N \rightarrow +\infty$$

In particular, at points of continuity  $S f(x) = f(x)$

### 3.7 Pointwise convergence of Fourier Series:

Recall: examining the question of when

$$S_N f(x) \rightarrow f(x)$$

where

$$S_N f(x) = \sum_{k=-N}^N \hat{f}(k) e^{ikx} \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

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Thm 1: Let  $f: \mathbb{R} \rightarrow \mathbb{C}$  be  $2\pi$ -periodic and piecewise regular

Then  $\forall x_0 \in \mathbb{R}$

$$S_N f(x_0) \xrightarrow{N \rightarrow \infty} \frac{f(x_0^+) + f(x_0^-)}{2} = f^*(x_0)$$

In particular  $S_N f(x_0) \rightarrow f(x_0)$  at each point of continuity of  $f$

Recall:  $f$  piecewise regular means

- (1) at most a finite # of discontinuities in  $Q = [-\pi, \pi]$ ,
- (2) at points  $\{x_j\}_{j=1}^m \subset Q$  where  $f(x_j^\pm)$  are finite
- (3)  $f'$  exists in  $Q \setminus S$  with  $S$  finite and  $f'$  is bounded and continuous on  $Q \setminus S$

In particular  $\frac{f(x) - f(x_0^\pm)}{x - x_0}$  are bounded  $\forall x_0 \in Q, x \neq x_0$

Ex:  $f \in C^1(\mathbb{R}, \mathbb{C})$  with uniform and absolute convergence per

N.B. 1. Theo 1 OK in the real case { $1, \cos nx, \sin nx$ }

2. If  $f \in BV(Q)$  would be OK

[ $f = f_1 - f_2$  with  $f_j \geq 0$  : Dirichlet proved for monotone + ]

Final Exam

give  
further  
fine  
lets do

Remark 3: Since  $f$  has at most a finite number of discontinuities and  $\hat{f}(k) = \frac{1}{2\pi} \int_Q e^{-ikx} f(x) dx$ ,

we can change  $f$  to  $f^{\#}(x) = \frac{f(x^+) + f(x^-)}{2}$  to say

$$Sf(x) = Sf^{\#}(x) = f^{\#}(x) \quad \forall x \in \mathbb{R}$$

Proof: (of Thm 1) We want  $S_N f(x_0) \rightarrow f^{\#}(x_0) \quad \forall x_0 \in \mathbb{R}$

1. WLOG  $x_0 = 0$  just consider a translate of  $f$  which will still be  $2\pi$ -periodic and piecewise regular

2. Express  $S_N f$  in terms of the Dirichlet kernel

$$\begin{aligned} S_N f(x) &= \sum_{k=-N}^N e^{ikx} \left( \int_Q e^{-iky} f(y) \frac{dy}{2\pi} \right) \\ &= \int_Q f(y) \left[ \sum_{k=-N}^N e^{ik(x-y)} \right] \frac{dy}{2\pi} \\ &\stackrel{?}{=} \int_Q f(y) D_N(x-y) \frac{dy}{2\pi} \\ &= f * b_N(x) \quad [\text{measure } = \frac{dy}{2\pi}] \end{aligned}$$

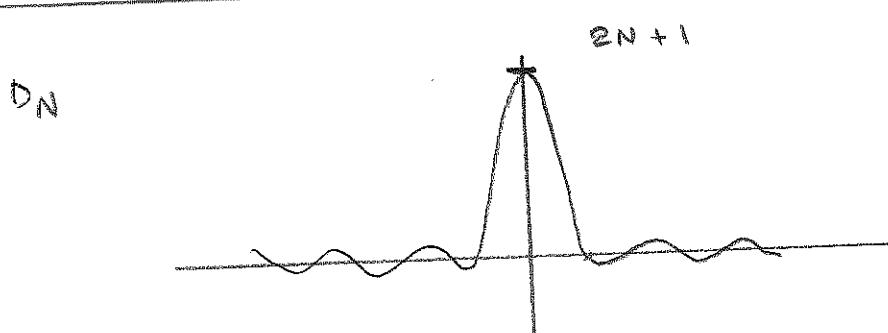
where  $D_N(x) = \sum_{k=-N}^N e^{ikx}$  is the Dirichlet kernel

Lemma 1: (Properties of  $D_N$ )

(a)  $D_N$  is real valued and even

$$(b) D_N(x) = \frac{\sin\left(\frac{(2N+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$(c) \frac{1}{\pi} \int_{-\pi}^{\pi} D_N(x) dx = \frac{1}{\pi} \int_0^\pi D_N(x) dx = 1$$



3 Given lemma 1:

$$\begin{aligned}
 S_N f(0) - f^*(0) &= \int_Q f(y) D_N(-y) \frac{dy}{2\pi} = f^*(0) \\
 &= \frac{1}{2\pi} \int_Q f(y) D_N(y) \frac{dy}{2\pi} = f^*(0) \text{ (use (a)} \\
 &\quad \text{by even}} \\
 &= \frac{1}{2} \left[ \frac{1}{\pi} \int_{-\pi}^0 f(y) D_N(y) dy + \frac{1}{\pi} \int_0^\pi f(y) D_N(y) dy \right] = f(0^+) - f(0^-) \\
 &= \frac{1}{2} \left[ \frac{1}{\pi} \int_{-\pi}^0 (f(y) - f(0^-)) D_N(y) dy + \frac{1}{\pi} \int_0^\pi (f(y) - f(0^+)) D_N(y) dy \right] \\
 &\quad \text{Lemma 1(c)} \\
 &= \frac{1}{2\pi} \int_Q \sin\left(\frac{(2N+1)y}{2}\right) g(y) dy
 \end{aligned}$$

with  $g(y) = \begin{cases} \frac{f(y) - f(0^-)}{\sin(\gamma/2)} & -\pi < y < 0 \\ 0 & y = 0 \\ \frac{f(y) - f(0^+)}{\sin(\gamma/2)} & 0 < y < \pi \end{cases}$

$$\bullet \text{ Notice } \sin((N+\frac{1}{2})y) = \underbrace{\sin Ny \cos \frac{y}{2}}_{u(y)} - \underbrace{\cos Ny \sin \frac{y}{2}}_{v(y)}$$

so

$$g(y) \sin((N+\frac{1}{2})y) = \underbrace{\sin Ny}_{u(y)} \underbrace{\cos \frac{y}{2}}_{v(y)} - \underbrace{\cos Ny}_{v(y)} \underbrace{\sin \frac{y}{2}}_{u(y)}$$

and

$$S_N f(0) - f^*(0) = \frac{1}{2\pi} \int_Q u(y) \sin Ny dy - \frac{1}{2\pi} \int_Q v(y) \cos Ny dy$$

$$\boxed{S_N f(0) - f^*(0) = \frac{1}{2} b_N(u) + \frac{1}{2} a_N(v)}$$

a sum of the sin, cos Fourier coeffs of  $u, v$  where

$$\begin{cases} a_N(v) = \hat{v}(N) + \hat{v}(-N) \\ b_N(u) = i(\hat{u}(N) - \hat{u}(-N)) \end{cases}$$

$$\bullet \text{ Hence, we need only } \boxed{\hat{u}(k), \hat{v}(k) \rightarrow 0 \text{ as } k \rightarrow \pm\infty}$$

This is true because  $u, v \in L^1(Q)$

Lemma 2: (Riemann-Lebesgue) For  $u \in L^1(\mathbb{R})$

$$\hat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} u(x) dx \rightarrow 0 \text{ as } \xi \rightarrow \pm\infty$$

here  $\hat{u}$  is the Fourier Transform of  $u$

\* Given Lemma 2, notice that  $u, v \in L^1(\mathbb{Q})$  since  $f$  is piecewise regular. Only problem is in  $y=0$  due to

$$\frac{f(y) - f(0^+)}{y - 0^+} \approx \text{difference quotient in } f \text{ at } 0$$

But  $f$  has bounded difference quotients

Hence  $u, v$  bounded on  $\mathbb{Q}$ , extend by zero to  $\mathbb{R}$

$$\begin{cases} \hat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{Q}} e^{ikx} u(x) dx \rightarrow 0 & \text{as } k \rightarrow \pm\infty \\ \hat{v}(k) \rightarrow 0 & \end{cases} \quad \frac{1}{2\pi} \int_{\mathbb{R}} x_Q u e^{-ikx} dx \quad f = \chi_Q u \in L^1(\mathbb{R})$$

□

We still need to check Lemma 1 (Properties of  $D_n$ ) exercise  
and Lemma 2

### Proof of Lemma 2

For each  $\xi \in \mathbb{R} \setminus \{0\}$  and char.  $\chi_{[a,b]}$  one has

$$\int_{\mathbb{R}} e^{-i\xi x} \chi_{[a,b]}(x) dx = \int_a^b e^{-i\xi x} dx = \frac{e^{-i\xi b} - e^{-i\xi a}}{-i\xi}$$

$$\left| \int_{\mathbb{R}} e^{-i\xi x} \chi_{[a,b]}(x) dx \right| \leq \frac{2}{|\xi|} \rightarrow 0 \text{ as } |\xi| \rightarrow +\infty$$

This extends to all finite linear combinations of char. functions of intervals, which is a dense family in  $L^1(\mathbb{R})$

$$\forall \varepsilon > 0 \quad \exists \quad h = \sum_{k=1}^N \chi_{[a_k, b_k]} \quad \text{st. } \|u - h\|_{L^1(\mathbb{R})} < \varepsilon$$

$$|\hat{u}(s)| = |\hat{u}(s) - \hat{h}(s) + \hat{h}(s)|$$

$$\leq \frac{1}{2\pi} \int_R |e^{-isx} (u(x) - h(x))| dx + |\hat{h}(s)|$$

$$< \frac{1}{2\pi} \varepsilon + |\hat{h}(s)| < \frac{\varepsilon}{\pi} \quad \text{if } |s| \text{ large}$$

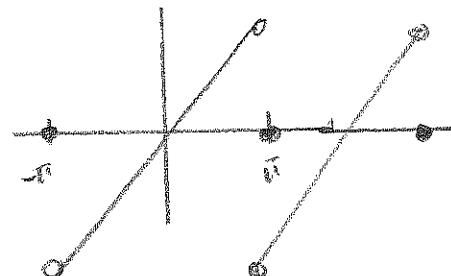
□



Remark 4: Fourier series expansions give rise to "new" summation formulas

e.g. + from last time with

$$\begin{cases} f(x) = x & x \in (-\pi, \pi) \\ f(-\pi) = 0 = f(\pi) \end{cases}$$



\* f is real and odd so

$$\begin{aligned} S f(x) &= \sum_{n=1}^{+\infty} b_n(f) \sin nx \quad \text{with } b_n(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx \\ &= f(x) \quad x \in [-\pi, \pi] \end{aligned}$$

IBP  $= \frac{2}{n} (-1)^{n+1}$

Now Parseval (Real Form)

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=1}^{+\infty} |b_n(f)|^2$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{+\infty} \frac{4}{n^2}$$

$$\frac{2}{\pi} \cdot \frac{x^3}{3} \Big|_0^\pi = \frac{2\pi^2}{3}$$

$$\Rightarrow \boxed{\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Remark 4: Other summability kernels yield different modes of interpreting  $Sf(x) = f(x)$

where  $S_N f(x) = f * D_N(x) = \int_{\mathbb{Q}} f(y) D_N(x-y) \frac{dy}{2\pi}$

Def 1: A family of kernels  $\{K_N\}_{N=1}^{+\infty}$  is a family of good kernels if

$$(i) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1 \quad \forall N \in \mathbb{N}$$

$$(ii) \exists M_0: \int_{-\pi}^{\pi} |K_N(x)| dx \leq M \quad \forall N \in \mathbb{N}$$

$$(iii) \forall \delta > 0 \quad \left( \int_{-\pi}^{\pi} |K_N(x)| dx \rightarrow 0 \text{ as } N \rightarrow +\infty \right)$$

Thm 2: If  $\{K_N\}_{N=1}^{+\infty}$  is a family of good kernels, then  $\forall f \in L^1(\mathbb{Q})$

$f * K_N(x) \rightarrow f(x)$  at each point of continuity of  $f$   
with uniform convergence if  $f \in C(\bar{\mathbb{Q}})$

proof: Very much like what we did in approx in  $L^p(E)$

(cf. [SS] Ch. 2)

Remark 5:  $\{D_N\}_{N=1}^{+\infty}$  FAILS to be a good family

because  $\int_{-\pi}^{\pi} |D_N(x)| dx \geq c \log(N) \text{ as } N \rightarrow +\infty$

so (ii) fails

Question What are some useful families of good kernels?

Ex 1: (Fejér kernels)

$$F_N(x) = \frac{D_0(x) + \dots + D_{N-1}(x)}{N} = \frac{1}{N} \frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$$

=  $\sum_{n=0}^{N-1} \cos(nx)$

Remark 6: Fejér kernels give rise to Cesàro summability of Fourier series

Def 2: A series of complex numbers  $\sum_{k=0}^{+\infty} c_k$  is said

to be Cesàro summable with Cesàro sum  $\sigma$  if

$$\sigma_N = \frac{S_0 + S_1 + \dots + S_{N-1}}{N} \rightarrow \sigma \quad S_n = \sum_{k=0}^n c_k$$

(that is, we average the first  $N$  partial sums)

Ex:  $\sum_{k=0}^{+\infty} (-1)^k$  has  $S_n = 1$   $n$  even  
 $\qquad\qquad\qquad (1 - 1 + 1 - 1 \dots) = 0$   $n$  odd

$$\sigma_{2k} = \frac{1}{2} \quad \forall k \in \mathbb{N}$$

$$\sigma_{2k+1} = \frac{k+1}{2k+1} \rightarrow \frac{1}{2} \quad k \rightarrow +\infty \Rightarrow \sigma = \frac{1}{2}$$

A more general notion of convergence

Thm 3: (Fejér) If  $f \in L^1(\mathbb{T}, \mathbb{C})$  then  $\sum_{k \in \mathbb{Z}} f(k) e^{ikx}$   
 is Cesàro summable with Cesàro sum  $= f(x)$  at each point of continuity, with uniform Cesàro summability  
 wif  $f \in C(\mathbb{T}, \mathbb{C})$

v. Chap 2 of [SS].