An Orlicz Space Approach to Superlinear Elliptic Systems *

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Abstract

In this paper we study superlinear elliptic systems in Hamiltonian form. Using an Orlicz-space setting, we extend the notion of critical growth to superlinear nonlinearities which do not have a polynomial growth. Existence of nontrivial solutions is proved for superlinear nonlinearities which are subcritical in this generalized sense.

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1 Introduction

In this paper we study nonlinear elliptic systems in Hamiltonian form:

$$\left\{ \begin{array}{ll} -\Delta u = g(v) & \text{in } \Omega \\ -\Delta v = f(u) & \text{in } \Omega \\ v > 0, u > 0 & \text{in } \Omega \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{array} \right. \quad (1.1)$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 3$), with smooth boundary $\partial\Omega$ and Δ is the Laplace operator.

For the scalar equation $-\Delta u = f(u)$ critical growth is given by $f(s) \sim s^{\frac{N+2}{N-2}}$. This is obtained by considering the related functional $\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} F(u)$, where $F(s) = \int_0^s f(t)dt$. The natural space for the first term is $H_0^1(\Omega)$, and then the maximal growth allowed for $F(s)$ is given by the Sobolev embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$. For the system (1.1) the associated functional is

$$\int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} (F(u) + G(v)) \quad (1.2)$$

If we consider this functional on $H_0^1(\Omega) \times H_0^1(\Omega)$, then we find again the maximal growths $F(s) \sim |s|^{2^*}$ and $G(s) \sim |s|^{2^*}$. However, in interesting papers by Hulshof-Van der

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Vorst [3] and Felmer-de Figueiredo [5] the use of Sobolev spaces of fractional order has been proposed. Roughly speaking, these spaces, denoted by $H^s(\Omega)$, $s > 0$, consist of the functions whose derivative of order s is in $L^2(\Omega)$ (these spaces can be defined via interpolation or via Fourier expansion). Introducing suitable selfadjoint operators $A^s : H^s(\Omega) \rightarrow L^2(\Omega)$, the first term in the functional (1.2) can be substituted by

$$\int_{\Omega} A^s u A^t v \quad , \quad \text{with } s + t = 2.$$

The maximal growth condition on $F(s) \sim |s|^{p+1}$ and $G(s) \sim |s|^{q+1}$ is then given by the largest values p and q such that $H^s \subset L^{p+1}$ and $H^t \subset L^{q+1}$. This yields the so-called *critical hyperbola*

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \quad (1.3)$$

One notes that now one of the nonlinearities may have a larger growth than $|s|^{2^*}$ provided the other nonlinearity has a suitably lower growth.

We propose here another approach: in order to have the term $\int_{\Omega} \nabla u \nabla v$ well-defined, we can use Hölder's inequality to estimate

$$|\int_{\Omega} \nabla u \nabla v| \leq \|u\|_{W_0^{1,\alpha}} \|v\|_{W_0^{1,\beta}} \quad , \quad \text{with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \quad (1.4)$$

and hence the first term in (1.2) can be considered on the space $W_0^{1,\alpha}(\Omega) \times W_0^{1,\beta}(\Omega)$. Again, the maximal growth for F and G is then given by the embeddings

$$W_0^{1,\alpha}(\Omega) \subset L^{p+1}(\Omega) \quad \text{and} \quad W_0^{1,\beta}(\Omega) \subset L^{q+1}(\Omega)$$

with

$$\frac{1}{p+1} = \frac{1}{\alpha} - \frac{1}{N} \quad \text{and} \quad \frac{1}{q+1} = \frac{1}{\beta} - \frac{1}{N} ; \quad (1.5)$$

thus, this approach yields the same critical hyperbola.

Consequently, the same equations studied in [3] and [5] can be treated also by this approach. However the second approach has the advantage that it can be generalized to more general situations by using an *Orlicz space* setting. Namely, we can replace the spaces $W_0^{1,\alpha}$ and $W_0^{1,\beta}$ by Sobolev-Orlicz spaces $W_0^1 L_A$ and $W_0^1 L_{\tilde{A}}$; here $W_0^1 L_A$ is given by the functions u such that $\int_{\Omega} A(|\nabla u|) < \infty$, where A is a so-called N-function, and u vanishes in a certain sense at the boundary. The function $\tilde{A}$ is the Young-conjugate function to A (see below). Then we have again a Hölder type inequality

$$|\int_{\Omega} \nabla u \nabla v| \leq 2 \|u\|_{W_0^1 L_A} \|v\|_{W_0^1 L_{\tilde{A}}}$$

The maximal growth for F and G are then determined by the Orlicz-space embeddings

$$W_0^1 L_A(\Omega) \subset L_{\Phi}(\Omega) \quad \text{and} \quad W_0^1 L_{\tilde{A}}(\Omega) \subset L_{\Psi}(\Omega) \quad (1.6)$$

where L_{Φ} and L_{Ψ} are Orlicz-spaces with suitable N-functions Φ and Ψ . In this way we can treat nonlinearities f and g in equation (1.1) that cannot be handled by the Hilbert space approach of [3] and [5].

Let us call two N-functions (Φ, Ψ) a *critical Orlicz pair*, if they have the maximal possible growth in the above embeddings (1.6), for suitable N-functions A and $\tilde{A}$. For example, $\Phi(s) = |s|^{p+1}$ and $\Psi(s) = |s|^{q+1}$ with $p, q \in (1, +\infty)$ satisfying (1.3) constitute a critical Orlicz pair, with $A(s) = |s|^\alpha$ and $\tilde{A}(s) = |s|^\beta$, where α and β are given by (1.5) (see Example 2.1).

We will prove

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain. Let $\Phi \in C^1$ be a given N-function, and set $\varphi(t) = \Phi'(t)$. Assume that*

$$\lim_{s \rightarrow \infty} \frac{\varphi(s) s}{\Phi(s)} = \theta_\Phi > \frac{N}{N-2}. \quad (1.7)$$

Then there exists an associate N-function Ψ such that (Φ, Ψ) form a critical Orlicz pair. Furthermore, the limit

$$\lim_{s \rightarrow \infty} \frac{s \Psi(s)}{\Psi(s)} = \theta_\Psi$$

exists, and

$$\frac{1}{\theta_\Phi} + \frac{1}{\theta_\Psi} = 1 - \frac{2}{N}. \quad (1.8)$$

We give some examples of critical Orlicz pairs:

Example 1.1 *Let*

$$\Phi(s) \sim |s|^{p+1} (\log |s|)^\alpha \quad \text{and} \quad \Psi(s) \sim |s|^{q+1} (\log |s|)^{-\alpha \frac{q+1}{p+1}},$$

with $\alpha > 0$ and $p, q \in (1, +\infty)$ satisfying (1.3). Then Φ and Ψ satisfy (1.7), with $\theta_\Phi = p+1$ and $\theta_\Psi = q+1$, respectively, and (Φ, Ψ) form a critical Orlicz-pair. (see the proof in Section 6).

Remark 1.2 *The restriction $\theta_\Phi > N/(N-2)$ is necessary in order to obtain a Ψ which is θ -regular, in the sense of Definition 2.10. Also in the polynomial case such a restriction, which here is $p+1 > N/(N-2)$, is necessary in order to obtain $q > 1$.*

Based on this characterization, we will prove an existence theorem for superlinear nonlinearities, which have subcritical growth with respect to the above specified Orlicz criticality.

We make the following hypotheses

- (H1) let $f, g \in C(\mathbb{R})$ and let F, G denote their primitives;
- (H2) there exist constants $\theta > 2$ and $t_0 > 0$ such that, for all $t \geq t_0$,

$$0 < \theta F(t) \leq t f(t) \quad \text{and} \quad 0 < \theta G(t) \leq t g(t).$$

- (H3) F and G are uniformly superquadratic near zero (for the definition, see Definition 5.1 below).

Then we prove

Theorem 1.3 *Suppose that (Φ, Ψ) is a critical Orlicz-pair. Suppose that F and G satisfy (H1) – (H3), and that F and G have an essentially slower growth than Φ and Ψ , respectively (see section 2 below). Suppose also that*

$$\lim_{t \rightarrow 0} \frac{F(t)}{\Phi(t)} = C_F < \infty \text{ and } \lim_{t \rightarrow 0} \frac{G(t)}{\Psi(t)} = C_G < \infty.$$

Then system (1.1) has a nontrivial solution.

Example 1.2 *Let (Φ, Ψ) denote the critical Orlicz pair given in Example 1.1. Suppose that $F(s) \sim s^{p+1}(\log s)^\beta$ and $G(s) \sim s^{q+1}(\log s)^{-\gamma}$, for s positive and large with $\beta < \alpha$ and $\gamma > \alpha \frac{q+1}{p+1}$. Then F and G have essentially slower growth than Φ and Ψ , respectively.*

2 Orlicz spaces

Here we recall some basic facts about Orlicz spaces, for more details see for instance [1, 4, 8]. Let M be a N-function, that is, $M : \mathbb{R} \rightarrow [0, +\infty)$ is continuous, convex, even, $M(t) = 0$ if and only if $t = 0$,

$$M(t)/t \rightarrow 0 \text{ as } t \rightarrow 0 \text{ and } M(t)/t \rightarrow +\infty \text{ as } t \rightarrow +\infty.$$

We say that an N-function A *dominates* an N-function B (near infinity) if, for some positive constant k , $B(x) \leq A(kx)$ (for $x \geq x_0$), and write $B \prec A$. A and B are *equivalent* if A dominates B and B dominates A ; then we write $A \sim B$. Finally, we say that B *increases essentially more slowly* than A if $\lim_{t \rightarrow \infty} \frac{B(kt)}{A(t)} = 0$, for all $k > 0$; in this case we write $B \prec\prec A$.

Associated to the N-function M we have the following class of functions.

Definition 2.1 (Orlicz class) *The Orlicz class is defined by*

$$K_M(\Omega) := \{u : \Omega \rightarrow \mathbb{R} : u \text{ measurable and } \int_{\Omega} M(u(x)) \, dx < \infty\}.$$

Orlicz classes are convex sets, but in general not linear spaces. One then defines

Definition 2.2 (Orlicz space) *The vector space $L_M(\Omega)$ generated by $K_M(\Omega)$ is called Orlicz space.*

Fact 1: The Orlicz class $K_M(\Omega)$ is a vector space, and hence equal to $L_M(\Omega)$ if and only if M satisfies the following

Definition 2.3 (Δ_2 -condition) *There exist numbers $k > 1$ and $t_0 \geq 0$ such that*

$$M(2t) \leq kM(t) \text{ , for } t \geq t_0$$

Furthermore, we define

Definition 2.4 (∇_2 -condition) *There exist numbers $h > 1$ and $t_1 \geq 0$ such that*

$$M(t) \leq \frac{1}{2h} M(ht) \text{ , for } t \geq t_1$$

We call a function satisfying the Δ_2 - and the ∇_2 - condition **Δ -regular**.

We remark that the Orlicz class depends only on the asymptotic growth of the function M ; therefore, also the Δ_2 -condition and the ∇_2 -condition need to be satisfied only near infinity.

We define the following norm on $L_M(\Omega)$:

Definition 2.5 (Luxemburg norm)

$$\|u\|_{(M)} = \inf\{ \lambda > 0 : \int_{\Omega} M(\frac{|u|}{\lambda}) \leq 1 \}$$

Fact 2: $(L_M, \|\cdot\|_{(M)})$ is a Banach space.

Definition 2.6 (Conjugate function) *Let*

$$\widetilde{M}(x) = \sup_{y>0} \{xy - M(y)\}$$

$\widetilde{M}$ is called the conjugate function of M .

It is clear that $\widetilde{\widetilde{M}} = M$, and M and $\widetilde{M}$ satisfy the **Young inequality**:

$$st \leq M(t) + \widetilde{M}(s) \quad \forall s, t \in \mathbb{R},$$

with **equality** when $s = M'(t)$ or $t = \widetilde{M}'(s)$. In the spaces L_M and $L_{\widetilde{M}}$ the **Hölder inequality** holds:

$$|\int_{\Omega} u(x)v(x)dx| \leq 2\|u\|_{(M)}\|v\|_{(\widetilde{M})}$$

Hence, for every $\widetilde{u} \in L_{\widetilde{M}}$ we can define a continuous linear functional $l_{\widetilde{u}}v := \int_{\Omega} \widetilde{u}vdx$ and $l_{\widetilde{u}} \in (L_M)^*$. Then we can define

$$\|\widetilde{u}\|_{\widetilde{M}} = \|l_{\widetilde{u}}\| = \sup_{\|v\|_{(M)} \leq 1} \int_{\Omega} \widetilde{u}(x)v(x)dx$$

Definition 2.7 $\|\widetilde{u}\|_{\widetilde{M}}$ is called the **Orlicz norm** on the space $L_{\widetilde{M}}$, and analogously one defines the Orlicz norm $\|u\|_M$ on L_M .

Thus, we have two different norms on L_M , the Luxemburg (or gauge) norm $\|\cdot\|_{(M)}$ and the Orlicz norm $\|\cdot\|_M$; they are equivalent, and satisfy

$$\|u\|_{(M)} \leq \|u\|_M \leq 2\|u\|_{(M)}. \quad (2.9)$$

In order to be precise about which norm is considered in the spaces, we are going to use from now on the following notations:

$$(L_M, \|\cdot\|_M) := L_M \text{ and } (L_M, \|\cdot\|_{(M)}) := L_{(M)},$$

and similarly for $\widetilde{M}$.

Fact 3: It follows from the definition of Orlicz norm that, if $u \in L_M$ and $\tilde{w} \in L_{\tilde{M}}$, then one has the following Hölder inequality

$$\left| \int_{\Omega} u \tilde{w} \, dx \right| \leq \|u\|_M \|\tilde{w}\|_{(\tilde{M})}. \quad (2.10)$$

Fact 4: L_M is **reflexive** if and only if M and $\tilde{M}$ satisfy the Δ_2 -condition, and then

$$(L_{(M)})^* = L_{\tilde{M}} \quad \text{and} \quad (L_{(\tilde{M})})^* = L_M$$

(see J.P. Gossez [2], Rao-Ren [8], p. 111)

Fact 5: If Φ is Δ -regular, then there exists a $\Phi_1 \sim \Phi$ such that $L_{\Phi} = L_{\Phi_1}$ as sets, and their Luxemburg norms (respectively Orlicz norms) are equivalent, with the following additional structure:

a) L_{Φ} and L_{Φ_1} are isomorphic, and both are reflexive spaces

b) L_{Φ_1} is uniformly convex and uniformly smooth

(see Rao-Ren [8], Thm 2, p. 297).

Next, we define the **Orlicz-Sobolev** spaces: Let A be a N-function. Then set

Definition 2.8

$$W^1 L_A = \left\{ u : \Omega \rightarrow \mathbb{R} ; \max_{|\alpha| \in \{0,1\}} \int_{\Omega} A(|D^{\alpha} u|) < +\infty \right\}$$

with Luxemburg norm

$$\|u\|_{W^1 L_A} := \max \{ \|D^{\alpha} u\|_{(M)} : |\alpha| \in \{0,1\} \}$$

On the space $W_0^1 L_{(A)}$, i.e. the space of functions in $W^1 L_A$ which vanish on the boundary, an equivalent Luxemburg norm is given by

$$\|u\|_{1,(A)} = \|\nabla u\|_{(A)} = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{|\nabla u|}{\lambda}\right) \leq 1 \right\}.$$

The equivalence of these two norms is a consequence of the Poincaré inequality,

$$\|u\|_{(M)} \leq C \sum_{i=1}^n \|D_i u\|_{(M)}, \quad \forall u \in W_0^1 L_M(\Omega),$$

(see [2]). In analogy with the above definition of the Orlicz norm in L_M we can define an Orlicz norm in $W_0^1 L_{(A)}$ by

$$\|u\|_{1,A} := \sup \left\{ \int_{\Omega} \nabla u \nabla \tilde{w} \, dx : \tilde{w} \in W_0^1 L_{(\tilde{A})}, \|\tilde{w}\|_{1,(\tilde{A})} \leq 1 \right\}$$

The space $W_0^1 L_{(A)}$ endowed with this new norm is denoted by $W_0^1 L_A$.

Definition 2.9 (Sobolev conjugate) (Adams [1], p. 248)

Suppose that $\int_1^{\infty} \frac{A^{-1}(t)}{t^{1+1/n}} dt = +\infty$. Then the Sobolev conjugate function $\Phi(t)$ is given by

$$\Phi^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{1+1/n}} d\tau, \quad t \geq 0.$$

Fact 6: Let Ω be bounded, and satisfying the cone property. Then

$$W^1 L_A(\Omega) \hookrightarrow L_\Phi(\Omega) \text{ continuously,}$$

and compactly into $L_G(\Omega)$, where G is any N -function increasing *essentially more slowly* than Φ , i.e. $\lim_{t \rightarrow \infty} \frac{G(kt)}{\Phi(t)} = 0$, for all $k > 0$.

Example 2.1 One easily checks that for $\Phi(s) = s^{p+1}$ the above formula yields $A(s) = cs^\alpha$, with α satisfying $\frac{1}{\alpha} = \frac{1}{p+1} + \frac{1}{N}$, i.e. we have the classical Sobolev imbedding theorem for $W^{1,\alpha}(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

Next, we make the following

Definition 2.10 Let $g \in C(\mathbb{R})$ be an N -function, and G its primitive. Then we say that G is θ - regular, if there exists a constant $\theta_G > 1$ such that

$$\lim_{s \rightarrow \infty} \frac{s g(s)}{G(s)} = \theta_G . \quad (2.11)$$

Let $F(t) = G^{-1}(t)$, and $f(t) = F'(t)$. Then the above condition is equivalent to

$$\lim_{t \rightarrow \infty} \frac{t f(t)}{F(t)} = \frac{1}{\theta_G} . \quad (2.12)$$

Indeed, we have $G(s) = t \Leftrightarrow F(t) = s$, and $f(t) = \frac{d}{dt}[G^{-1}(t)] = \frac{1}{g(s)}$.

Note that by (Rao-Ren [8], p. 26) we have

Fact 7: If G is θ - regular, then G is Δ - regular, i.e. $G \in \Delta_2 \cap \nabla_2$.

3 Orlicz space criticality

Definition 3.1 (critical Orlicz pair) Let Φ and Ψ be Δ -regular N -functions. Then (Φ, Ψ) are a **critical Orlicz pair** if there exist Δ -regular and conjugate N -functions A and $\tilde{A}$ such that L_Φ and L_Ψ are the smallest Orlicz spaces with

$$W^1 L_A \hookrightarrow L_\Phi \quad , \quad W^1 L_{\tilde{A}} \hookrightarrow L_\Psi .$$

Consider the following example:

Example 3.1 In Example 2.1 we saw that to $\Phi(s) = s^{p+1}$ corresponds the inverse Sobolev conjugate $A(s) = cs^\alpha$, with

$$\frac{1}{p+1} + \frac{1}{N} = \frac{1}{\alpha} .$$

The conjugate function $\tilde{A}$ to A is given by $\tilde{A}(s) = cs^\beta$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, which in turn has as Sobolev conjugate $\Psi(s) = s^{q+1}$, with

$$\frac{1}{q+1} + \frac{1}{N} = \frac{1}{\beta} .$$

Adding the two equation yields

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}.$$

This is the critical hyperbola, see de Figueiredo-Felmer [5] and Hulshoff-VanderVorst [3]. Thus, $(|s|^{p+1}, |s|^{q+1})$ are a critical Orlicz pair, and so the above theorem contains the critical hyperbola as a special case. We remark that the proof given here is also new in the polynomial case; in [5] and [3] fractional Sobolev spaces H^s were used in order to conserve the Hilbert space structure.

3.1 Proof of Theorem 1.1

1) Hypothesis (1.7) expresses the fact that the functions Φ is θ -regular with $\theta_\Phi > N/(N-2)$. Let A be the inverse Sobolev conjugate of Φ , see Definition 2.9. Note that $W^1 L_A$ is the largest Orlicz-Sobolev space that embeds into L_Φ .

Claim 1: A is θ - regular, with $\theta_A = \frac{N\theta_\Phi}{N+\theta_\Phi} > 1$.

Indeed, let $F(s) = \Phi^{-1}(s)$ and $B(t) = A^{-1}(t)$. Then $F(s) = \int_0^s \frac{B(t)}{t^{1+1/N}} dt$, and hence

$$f(s) = \frac{B(s)}{s^{1+1/N}}.$$

Then we have by (2.12)

$$\frac{1}{\theta_\Phi} = \lim_{s \rightarrow \infty} \frac{f(s)s}{F(s)} = \lim_{s \rightarrow \infty} \frac{B(s) s^{-1/N}}{F(s)}.$$

Then, by l'Hôpital's rule

$$\lim_{s \rightarrow \infty} \frac{B(s) s^{-1/N}}{F(s)} = \lim_{s \rightarrow \infty} \frac{b(s) s^{-1/N} - \frac{1}{N} s^{-\frac{1}{N}-1} B(s)}{\frac{B(s)}{s^{1+1/N}}} = \lim_{s \rightarrow \infty} \frac{b(s)s}{B(s)} - \frac{1}{N}.$$

We conclude that

$$\frac{1}{\theta_\Phi} = \lim_{s \rightarrow \infty} \frac{b(s)s}{B(s)} - \frac{1}{N}$$

and thus

$$\lim_{s \rightarrow \infty} \frac{b(s)s}{B(s)} = \frac{1}{\theta_\Phi} + \frac{1}{N} < 1.$$

This implies that A is θ - regular, with $\theta_A = \frac{N\theta_\Phi}{N+\theta_\Phi} > 1$.

2) Next, let $\tilde{A}$ be the conjugate function of A , given by definition 2.6. $\tilde{A}$ is a N -function, and Δ -regular, see Rao-Ren [8], Cor. 4, p. 26.

In the following, suppose that $s = A'(t)$ (iff $t = \tilde{A}'(s)$); note that $t \rightarrow \infty$ iff $s \rightarrow \infty$. Then

$$\frac{1}{\theta_A} = \lim_{t \rightarrow \infty} \frac{A(t)}{tA'(t)} = \lim_{t \rightarrow \infty} \frac{A(t)}{ts} = \lim_{s \rightarrow \infty} \frac{st - \tilde{A}(s)}{st} = 1 - \lim_{s \rightarrow \infty} \frac{\tilde{A}(s)}{s\tilde{A}'(s)} = 1 - \frac{1}{\theta_{\tilde{A}}}.$$

Thus, $\tilde{A}$ is θ - regular, with $\theta_{\tilde{A}} > 1$.

We can now define the corresponding Orlicz-Sobolev space $W^1L_{\tilde{A}}$.

3) Next, use definition 2.9 again to define the function Ψ ; by Adams [1], p. 248, Ψ is an N -function.

Claim 2: Ψ is θ - regular, with $\theta_\Psi = \frac{N\theta_{\tilde{A}}}{N-\theta_{\tilde{A}}}$.

This follows similarly as in claim 1, reversing the direction in the arguments.

Finally, L_Ψ is the smallest Orlicz space into which $W^1L_{\tilde{A}}$ imbeds continuously.

Thus, we have shown that (Φ, Ψ) is a critical Orlicz pair.

Finally, we have

$$\frac{1}{\theta_\Phi} + \frac{1}{\theta_\Psi} = \frac{N - \theta_A}{N\theta_A} + \frac{N - \theta_{\tilde{A}}}{N\tilde{A}} = \frac{1}{\theta_A} - \frac{1}{N} + \frac{1}{\theta_{\tilde{A}}} - \frac{1}{N} = 1 - \frac{2}{N}.$$

4 The tilde - map

In this section we define a map from $W_0^1L_A$ to space $W_0^1L_{(\tilde{A})}$, where $\tilde{A}$ is the Young conjugate of A .

Theorem 4.1 *For each $u \in W_0^1L_A$ consider*

$$S := \sup \left\{ \int_{\Omega} \nabla u(x) \nabla \tilde{w}(x) dx : \tilde{w} \in W_0^1L_{(\tilde{A})}, \|\tilde{w}\|_{1,(\tilde{A})} = \|u\|_{1,A} \right\}. \quad (4.13)$$

Then there exists a unique $\tilde{u} \in W_0^1L_{(\tilde{A})}$ such that

$$\|\tilde{u}\|_{1,(\tilde{A})} = \|u\|_{1,A} \text{ and } S = \int_{\Omega} \nabla u(x) \nabla \tilde{u}(x) dx = \|u\|_{1,A} \|\tilde{u}\|_{1,(\tilde{A})}$$

Furthermore, $\tilde{u}$ depends continuously (but nonlinearly) on u .

Proof. We first remark that by Fact 5 we may assume that all the spaces W^1L_A , $W^1L_{(A)}$, $W^1L_{\tilde{A}}$ and $W^1L_{(\tilde{A})}$ are reflexive and uniformly convex. Observe also that by (2.10)

$$S \leq \sup \|\tilde{w}\|_{1,(\tilde{A})} \|u\|_{1,A} = \|u\|_{1,A}^2.$$

- *Existence:* Let $\{\tilde{v}_n\}$ be a maximizing sequence for (4.13). Since the sequence is bounded, we have by reflexivity that (for a subsequence) $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $W_0^1L_{(\tilde{A})}$. So $\int_{\Omega} \nabla u \nabla \tilde{v}_n \rightarrow S$, and consequently $\int_{\Omega} \nabla u \nabla \tilde{v} = S$, that is, the supremum is attained. It remains to prove that $\|\tilde{u}\|_{1,(\tilde{A})} = \|u\|_{1,A}$. Suppose by contradiction that $\|\tilde{u}\|_{1,(\tilde{A})} = k < \|u\|_{1,A}$. Take $\tilde{w} = (\tilde{u}/k)\|u\|_{1,A}$. Then for this $\tilde{w}$ we have

$$S \geq \int_{\Omega} \nabla u \nabla \tilde{w} = \int_{\Omega} \nabla u \nabla \tilde{u} \frac{\|u\|_{1,A}}{k} > S,$$

which is impossible.

- *Uniqueness*: by the uniform convexity of $W_0^1 L_A$, $\tilde{u}$ is unique

- *Continuity*: Let $u_n \rightarrow u \neq 0$ in $W_0^1 L_A$. By the above we have $\|\tilde{u}_n\|_{1,(\tilde{A})} = \|u_n\|_{1,A}$ and $\|\tilde{u}\|_{1,(\tilde{A})} = \|u\|_{1,A}$. Consequently $\|\tilde{u}_n\|_{1,(\tilde{A})} \rightarrow \|\tilde{u}\|_{1,A}$. So we have that, for some subsequence, $\tilde{u}_n \rightharpoonup \tilde{v}$ in $W_0^1 L_{(\tilde{A})}$. If we prove that $\tilde{v} = \tilde{u}$, then in fact we are concluding that the whole sequence $\tilde{u}_n$ converges strongly to $\tilde{u}$ in $W_0^1 L_{(\tilde{A})}$. To this end observe that

$$\int_{\Omega} \nabla u_n \nabla \tilde{u}_n = \|\tilde{u}_n\|_{1,(\tilde{A})} \|u_n\|_{1,A}$$

implies

$$\int_{\Omega} \nabla u \nabla \tilde{v} = \|\tilde{u}\|_{1,(\tilde{A})} \|u\|_{1,A}.$$

We claim that $\|\tilde{v}\|_{1,(\tilde{A})} = \|u\|_{1,A}$, and then by the uniqueness of $\tilde{u}$ it follows that $\tilde{v} = \tilde{u}$ and the proof is complete. The claim is proved by contradiction as we did in the existence above assuming that $\|\tilde{v}\|_{1,(\tilde{A})} = k < \|u\|_{1,A}$. $\blacksquare$

Using the previous theorem we now define the "tilde-map"

$$\begin{array}{ccc} \sim & : & W_0^1 L_A \longrightarrow W_0^1 L_{(\tilde{A})} \\ & & u \longmapsto \tilde{u} \end{array}$$

which is continuous.

Remark 4.2 *It follows from the construction that the tilde-map is positively homogeneous, i.e.*

$$\widetilde{\rho u} = \rho \tilde{u}, \quad \forall u \in W^1 L_A, \quad \forall \rho \geq 0$$

With the help of the tilde-map, we define two continuous sub-manifolds of

$$E := W_0^1 L_A \times W_0^1 L_{(\tilde{A})}$$

by

$$E^+ = \{(u, \tilde{u}) ; u \in W_0^1 L_A\} \quad \text{and} \quad E^- = \{(u, -\tilde{u}) ; u \in W_0^1 L_A\}.$$

We remark that E^+ and E^- are nonlinear submanifolds of E when regarded with respect to the standard vector space structure of E . Surprisingly, E^+ and E^- turn out to be *linear* with respect to the following notion of *tilde-sum*:

Definition 4.3 (tilde sum) *Given elements $(u, \tilde{v}) \in E$ and $(y, \tilde{z}) \in E$, we set*

$$(u, \tilde{v}) \tilde{+} (y, \tilde{z}) := (u + y, \widetilde{v + z})$$

Indeed, with this notion we can prove:

Lemma 4.4

1) *Let $(u, \tilde{u}) \in E^+$ and $(v, \tilde{v}) \in E^+$; then, for all $\alpha, \beta \in \mathbb{R}$*

$$\alpha(u, \tilde{u}) \tilde{+} \beta(v, \tilde{v}) \in E^+ \quad \text{and} \quad \alpha(u, \tilde{u}) \tilde{+} \beta(v, \tilde{v}) = (\alpha u + \beta v, \widetilde{\alpha u + \beta v})$$

2) *For every $(y, \tilde{z}) \in E$ there exist unique elements $(u, \tilde{u}) \in E^+$ and $(v, -\tilde{v}) \in E^-$ such that*

$$(y, \tilde{z}) = (u, \tilde{u}) \tilde{+} (v, -\tilde{v}),$$

i.e. we can write

$$E = E^+ \tilde{\oplus} E^-.$$

Proof.

1) We have

$$\begin{aligned}\alpha(u, \tilde{v}) \tilde{+} \beta(y, \tilde{z}) &= (\alpha u, \alpha \tilde{v}) \tilde{+} (\beta y, \beta \tilde{z}) \\ &= (\alpha u, \widetilde{\alpha v}) \tilde{+} (\beta y, \widetilde{\beta z}) = (\alpha u + \beta y, \widetilde{\alpha v + \beta z})\end{aligned}$$

2) - *Uniqueness*: suppose that $(u + v, \widetilde{u - v}) = (u_1 + v_1, \widetilde{u_1 - v_1})$; then $u + v = u_1 + v_1$ and $\widetilde{u - v} = \widetilde{u_1 - v_1}$, which implies $u = u_1$ and $v = v_1$.

- *Existence*: we look for elements u and v in $W_0^1 L_A$ such that

$$(y, \tilde{z}) = (u, \tilde{u}) \tilde{+} (v, -\tilde{v}) = (u + v, \widetilde{u - v})$$

That is, $y = u + v$ and $z = u - v$, and hence $u = \frac{y+z}{2}$ and $v = \frac{y-z}{2}$. ■

5 An Existence Theorem

In this section we prove the existence of non-trivial solution for system (1.1), in the case of superlinear nonlinearities which have subcritical growth with respect to a given critical Orlicz pair.

5.1 The functional

In this section we define the framework for the functional I associated to problem (1.1) and given in (1.2).

We first give the precise definition of *uniformly superquadratic near zero*.

Definition 5.1 *A continuous function $H : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly superquadratic near zero, if there exist numbers $\sigma > 2$ and $c \geq 1$ such that*

$$H(st) \leq c s^\sigma H(t), \quad \forall t > 0, \quad \forall s \in [0, 1]$$

Note that if $H(t) = t^p$ with $p > 2$, then H satisfies the definition with $\sigma = p$.

In section 3 we have proved the existence of critical Orlicz pairs (Φ, Ψ) . As specified in Theorem 1.3, we assume that the functions F and G grow essentially more slowly than Φ and Ψ , respectively. Since we are interested in positive solutions we redefine F and G to be zero on $(-\infty, 0]$.

Consider the functional $I : W_0^1 L_A(\Omega) \times W_0^1 L_{(\tilde{A})}(\Omega) \rightarrow \mathbb{R}$ given by

$$I(u, \tilde{v}) = \int_{\Omega} \nabla u \nabla \tilde{v} \, dx - \int_{\Omega} [F(u) + G(\tilde{v})] \, dx \quad (5.14)$$

Here $\tilde{v} \in W_0^1 L_{(\tilde{A})}(\Omega)$ is an independent variable; we write $\tilde{v}$ to emphasize that $\tilde{v}$ belongs to the space $W_0^1 L_{(\tilde{A})}(\Omega)$

The functional I is well defined and belongs to the class C^1 with

$$I'(u, \tilde{v})(\eta, \tilde{\xi}) = \int_{\Omega} [\nabla u \nabla \tilde{\xi} + \nabla \tilde{v} \nabla \eta] \, dx - \int_{\Omega} [f(u)\eta + g(\tilde{v})\tilde{\xi}] \, dx, \quad (5.15)$$

for all $(\eta, \tilde{\xi}) \in W_0^1 L_A(\Omega) \times W_0^1 L_{(\tilde{A})}(\Omega)$. Consequently, critical points of the functional I correspond to the weak solutions of (1.1).

5.2 The Geometry of the Linking Theorem

We first prove that the functional I given in (5.14) has the geometry of the linking theorem.

Let E^+ and E^- be as in section 4.

Lemma 5.2 *There exist $\rho_0, \sigma_0 > 0$ such that $I(z) \geq \sigma_0$, for all $z \in \partial B_{\rho_0} \cap E^+$.*

Proof.

$$I(u, \tilde{u}) = \int_{\Omega} \nabla u \nabla \tilde{u} \, dx - \int_{\Omega} F(u) \, dx - \int_{\Omega} G(\tilde{u}) \, dx$$

Now, using that

$$\int_{\Omega} \nabla u \nabla \tilde{u} \, dx = \|u\|_{1,A} \|\tilde{u}\|_{1,(\tilde{A})} = \|u\|_{1,A}^2 = \|\tilde{u}\|_{1,(\tilde{A})}^2$$

and by (2.9)

$$\|u\|_{1,A} \geq \|u\|_{1,(A)}$$

we obtain that

$$I(u, \tilde{u}) \geq \frac{1}{2} \|u\|_{1,(A)}^2 - \int_{\Omega} F(u) \, dx + \frac{1}{2} \|\tilde{u}\|_{1,(\tilde{A})}^2 - \int_{\Omega} G(\tilde{u}) \, dx .$$

Assume that $\rho \in (0, 1)$, $u \in W_0^1 L_A(\Omega)$ and $\|u\|_{1,(A)} = c_1^{-1}$, where $c_1 > 0$ is such that

$$\|u\|_{(\Phi)} \leq c_1 \|u\|_{1,(A)} .$$

Since

$$\|u\|_{(\Phi)} = \inf \{ \lambda > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{\lambda}\right) \leq 1 \} \leq 1,$$

it follows that $\int_{\Omega} \Phi(|u|) \leq 1$, and thus $\int_{\Omega} F(|u|) \leq c$. By hypothesis (H3) we get for $0 \leq \rho \leq 1$

$$\int_{\Omega} F(\rho u) \, dx \leq c \rho^{\sigma} \int_{\Omega} F(u) \, dx \leq c \rho^{\sigma} .$$

Hence we obtain that

$$\frac{1}{2} \|\rho u\|_{1,(A)}^2 - \int_{\Omega} F(\rho u) \, dx \geq \frac{1}{2} \rho^2 \|u\|_{1,(A)}^2 - c \rho^{\sigma} .$$

Arguing similarly for G and $\tilde{u}$ we get

$$\frac{1}{2} \|\rho \tilde{u}\|_{1,(\tilde{A})}^2 - \int_{\Omega} G(\rho \tilde{u}) \, dx \geq \frac{1}{2} \rho^2 \|\tilde{u}\|_{1,(\tilde{A})}^2 - c \rho^{\sigma_1} .$$

By joining the two estimates we can find a $\rho_0 > 0$ such that

$$I(u, \tilde{u}) \geq \sigma_0 > 0 \quad \text{for } \|(u, \tilde{u})\| = \rho_0 > 0$$

This concludes the proof. ■

Let e_1 denote the first eigenfunction of the Laplacian, with $\|(e_1, \tilde{e}_1)\| = 1$ and set

$$Q = \{r(e_1, \tilde{e}_1) \mp w : w \in E^-, \|w\| \leq R_0 \text{ and } 0 \leq r \leq R_1\} .$$

Lemma 5.3 *There exist positive constants R_0, R_1 such that $I(z) \leq 0$ for all $z \in \partial Q$.*

Proof. Notice that the boundary ∂Q of the set Q is taken in the set $\mathbb{R}(e_1, \tilde{e}_1) \tilde{+} E^-$, and consists of three parts.

(i) If $z \in \partial Q \cap E^-$ we have $I(z) \leq 0$ because, for all $z = (\omega, -\tilde{\omega}) \in E^-$,

$$I(z) = -\|\omega\|_{1,(A)}^2 - \int_{\Omega} [F(\omega) + G(-\tilde{\omega})] dx \leq 0.$$

(ii) If $z = r(e_1, \tilde{e}_1) \tilde{+} (\omega, -\tilde{\omega}) = (re_1 + \omega, \widetilde{re_1 - \omega}) \in \partial Q$ with $\|(\omega, -\tilde{\omega})\| = R_0$ and $0 \leq r \leq R_1$, we proceed as follows:

First step: assume that $R_1 = 1$:

$$\begin{aligned} I(z) &\leq \int_{\Omega} \nabla(re_1 + \omega) \nabla(\widetilde{re_1 - \omega}) dx \\ &= - \int_{\Omega} \nabla(\omega - re_1) \nabla(\widetilde{\omega - re_1}) dx - 2r \int_{\Omega} \nabla e_1 \nabla(\widetilde{\omega - re_1}) \\ &\leq -\|\omega - re_1\|_{1,A}^2 + 2\|re_1\|_{1,A} \|\widetilde{\omega - re_1}\|_{1,(\tilde{A})} \\ &\leq -\|\omega - re_1\|_{1,A}^2 + 2\|re_1\|_{1,A} \|\omega - re_1\|_{1,A} \\ &\leq -\|\omega\|_{1,A}^2 - \|re_1\|_{1,A}^2 + 2\|re_1\|_{1,A} \|\omega\|_{1,A} + 2\|re_1\|_{1,A} (\|\omega\|_{1,A} + \|re_1\|_{1,A}) \\ &\leq -\|\omega\|_{1,A}^2 + 4r\|\omega\|_{1,A} + r^2 \end{aligned}$$

Since $2\|\omega\|_{1,A}^2 \geq \|\omega\|_{1,A}^2 + \|\tilde{\omega}\|_{1,(\tilde{A})}^2 = R_0^2$, we conclude that the last expression is ≤ 0 , for R_0 sufficiently large.

Second step: observe that using homogeneity this now holds for all $\rho \geq 1$ with $0 \leq r \leq \rho$ and $\|(\omega, -\tilde{\omega})\| = \rho R_0$.

(iii) Finally, let $z = \rho(e_1, \tilde{e}_1) \tilde{+} \rho(\omega, -\tilde{\omega}) = (\rho e_1 + \rho\omega, \widetilde{\rho e_1 - \rho\omega}) \in \partial Q$ with $\|e_1\|_{1,(A)} = 1/2$ and $\|(\omega, -\tilde{\omega})\| \leq R_0$.

We show: there exists $R_1 > 0$ sufficiently large such that for all $\rho \geq R_1$, we have $I(z) \leq 0$. We use that $W_0^1 L_A(\Omega) \hookrightarrow L_{\Phi}(\Omega)$, $W_0^1 L_{\tilde{A}}(\Omega) \hookrightarrow L_{\Psi}(\Omega)$, and that by assumption (H2): $F(s) \geq c|s|^{\theta} - c_1$ and $G(s) \geq c|s|^{\theta} - c_1$, for some $\theta > 2$; then

$$\begin{aligned} I(z) &= \int_{\Omega} \nabla(\rho e_1 + \rho\omega) \nabla(\widetilde{\rho e_1 - \rho\omega}) dx - \int_{\Omega} [F(\rho e_1 + \rho\omega) + G(\widetilde{\rho e_1 - \rho\omega})] dx \\ &\leq \rho^2 \|e_1 + \omega\|_{1,A} \|\widetilde{e_1 - \omega}\|_{1,(\tilde{A})} - c \int_{\Omega} |\rho e_1 + \rho\omega|^{\theta} dx + c_1 - c \int_{\Omega} |\widetilde{\rho e_1 - \rho\omega}|^{\theta} dx + c_1 \\ &\leq \rho^2 [\|e_1\|_{1,A} + \|\omega\|_{1,A}]^2 - c\rho^{\theta} \left\{ \int_{\Omega} |e_1 + \omega|^{\theta} dx + \int_{\Omega} |\widetilde{e_1 - \omega}|^{\theta} dx \right\} + 2c_1 \end{aligned}$$

Thus

$$I(z) \leq \rho^2(1 + R_0)^2 - c\rho^{\theta}\delta_0 + 2c_1, \quad (5.16)$$

where

$$\delta_0 := \inf_{\|(\omega, -\widetilde{\omega})\| \leq R_0} \left\{ \int_{\Omega} |e_1 + \omega|^\theta dx + \int_{\Omega} |\widetilde{e_1 - \omega}|^\theta dx \right\} > 0.$$

Indeed, suppose by contradiction that there exists a sequence $(\omega_n) \subset W_0^1 L_A(\Omega)$ such that $\|(\omega_n, -\widetilde{\omega}_n)\| \leq R_0$ and

$$\lim_{n \rightarrow \infty} \left\{ \int_{\Omega} |e_1 + \omega_n|^\theta dx + \int_{\Omega} |\widetilde{e_1 - \omega_n}|^\theta dx \right\} = 0.$$

Taking a subsequence, we may assume that $\omega_n \rightarrow \omega \in L^\theta$ (since $W^1 L_A \subset \subset L_F \subset L^\theta$) which implies that $e_1 + \omega_n \rightarrow e_1 + \omega$ and $\widetilde{e_1 - \omega_n} \rightarrow \widetilde{e_1 - \omega}$ in L^θ , where we have used the continuity of the tilde mapping. Thus, taking the limit we see that

$$\int_{\Omega} |e_1 + \omega|^\theta dx + \int_{\Omega} |\widetilde{e_1 - \omega}|^\theta dx = 0$$

which implies that $e_1 + \omega = \widetilde{e_1 - \omega} = 0$. So, $e_1 = \omega = 0$, which is a contradiction.

Finally, using (5.16) we can find $R_1 > 0$ such that $I(z) \leq 0$ for all $\rho \geq R_1$, and hence, the geometry of the linking theorem holds. $\blacksquare$

5.3 On Palais-Smale Sequences

Proposition 5.4 *Let $(u_m, \widetilde{v}_m) \in E$ such that*

- (I₁) $I(u_m, \widetilde{v}_m) = c + \delta_m$, where $\delta_m \rightarrow 0$ as $m \rightarrow +\infty$;
- (I₂) $|I'(u_m, \widetilde{v}_m)(\eta, \widetilde{\xi})| \leq \varepsilon_m \|(\eta, \widetilde{\xi})\|$, for $\eta, \xi \in \{u_m, v_m\}$, where $\varepsilon_m \rightarrow 0$ as $m \rightarrow +\infty$,

then

$$\begin{aligned} \|u_m\|_{1,A} &\leq C, \quad \|\widetilde{v}_m\|_{1,(\widetilde{A})} \leq C, \\ \int_{\Omega} f(u_m) u_m dx &\leq C, \quad \int_{\Omega} g(\widetilde{v}_m) \widetilde{v}_m dx \leq C \\ \int_{\Omega} F(u_m) dx &\leq C, \quad \int_{\Omega} G(\widetilde{v}_m) dx \leq C \end{aligned}$$

Proof. From (I₁) we have

$$\int_{\Omega} \nabla u_m \nabla \widetilde{v}_m dx - \int_{\Omega} F(u_m) dx - \int_{\Omega} G(\widetilde{v}_m) dx = c + \delta_m. \quad (5.17)$$

Taking $(\eta, \widetilde{\xi}) = (u_m, \widetilde{v}_m)$ in (I₂) we have

$$|2 \int_{\Omega} \nabla u_m \nabla \widetilde{v}_m dx - \int_{\Omega} f(u_m) u_m dx - \int_{\Omega} g(\widetilde{v}_m) \widetilde{v}_m dx| \leq \varepsilon_m \|(u_m, \widetilde{v}_m)\|, \quad (5.18)$$

which together with (I₁) and (H₃) implies that

$$(\theta - 2) \int_{\Omega} [F(u_m) + G(\widetilde{v}_m)] dx \leq 2c + 2\delta_m + \varepsilon_m \|(u_m, \widetilde{v}_m)\|.$$

Thus

$$\begin{aligned} \int_{\Omega} F(u_m) \, dx &\leq c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|) , \\ \int_{\Omega} G(\tilde{v}_m) \, dx &\leq c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|) \end{aligned}$$

and then by (5.17)

$$|\int_{\Omega} \nabla u_m \nabla \tilde{v}_m \, dx| \leq c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|)$$

and finally by (5.18) also

$$\begin{aligned} \int_{\Omega} f(u_m) u_m \, dx &\leq c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|) \\ \int_{\Omega} g(\tilde{v}_m) \tilde{v}_m \, dx &\leq c(1 + \delta_m + \varepsilon_m \|(u_m, v_m)\|) \end{aligned}$$

Taking $(\eta, \tilde{\xi}) = (0, \tilde{u}_m)$ in (I_2) we have

$$|\int_{\Omega} \nabla u_m \nabla \tilde{u}_m \, dx - \int_{\Omega} g(\tilde{v}_m) \tilde{u}_m \, dx| \leq \varepsilon_m \|(0, \tilde{u}_m)\| = \varepsilon_m \|\tilde{u}_m\|_{1,(\tilde{A})},$$

thus,

$$\|u_m\|_{1,A}^2 - \int_{\Omega} g(\tilde{v}_m) \tilde{u}_m \, dx \leq \varepsilon_m \|\tilde{u}_m\|_{1,(\tilde{A})} \quad (5.19)$$

Setting $\tilde{U}_m = \tilde{u}_m / C_0 \|\tilde{u}_m\|_{1,(\tilde{A})}$ and $V_m = v_m / C_1 \|v_m\|_{1,A}$ we have $\|\tilde{U}_m\|_{1,(\tilde{A})} = 1/C_0$ and $\|\tilde{U}_m\|_{(\Psi)} \leq C_0 \|\tilde{U}_m\|_{1,(\tilde{A})} \leq 1$ and thus by (5.19)

$$\|u_m\|_{1,A} \leq C_0 \int_{\Omega} g(\tilde{v}_m) \tilde{U}_m + \epsilon_m \quad (5.20)$$

Note also that

$$\frac{1}{C} \int_{\Omega} G(\tilde{U}_m) \leq \int_{\Omega} \Psi(\tilde{U}_m) \, dx \leq 1$$

since $\|\tilde{U}_m\|_{(\Psi)} = \inf\{\lambda : \int_{\Omega} \Psi(\frac{\tilde{U}_m}{\lambda}) \leq 1\}$.

We now rely on the following elementary inequalities

$$xy \leq F(x) + f^{-1}(y)y \quad \text{and} \quad xy \leq G(x) + g^{-1}(y)y \quad (5.21)$$

Applying (5.21) to the first term on the right in (5.20), with $y = g(\tilde{v}_m)$ and $x = \tilde{U}_m$ yields

$$\begin{aligned} \int_{\Omega} g(\tilde{v}_m) \tilde{U}_m \, dx &\leq \int_{\Omega} G(\tilde{U}_m) + \int_{\Omega} g(\tilde{v}_m) \tilde{v}_m \, dx \\ &\leq C(1 + \delta_m + \varepsilon_m \|(u_m, \tilde{v}_m)\|). \end{aligned}$$

Now using (5.20), we get

$$\|u_m\|_{1,A} \leq \varepsilon_m + C(1 + \delta_m + \varepsilon_m \|(u_m, \tilde{v}_m)\|)$$

Arguing similarly, choosing $(\eta, \tilde{\xi}) = (v_m, 0)$, yields

$$\|\tilde{v}_m\|_{1,(\tilde{A})} \leq \varepsilon_m + C(1 + \delta_m + \varepsilon_m \|(u_m, \tilde{v}_m)\|)$$

Joining the two estimates yields the claim. ■

5.4 Approximation by finite-dimensional problem

The functional I given by (1.2) is strongly indefinite near zero, since the first term is positive on the submanifold E^+ , and negative on the submanifold E^- . Since both E^+ and E^- are infinite dimensional, the standard linking theorems do not apply. We overcome this difficulty by using a finite-dimensional approximation. Denoting by $e_1, e_2, \dots$ an orthonormal basis of eigenfunctions associated to the eigenvalues $\lambda_1, \lambda_2, \dots$ of the Laplacian (with Dirichlet boundary conditions), we set $E_n = \text{span}\{e_1, e_2, \dots, e_n\}$. Let

$$E_n^+ := \{(z, \tilde{z}) ; z \in E_n\} , \quad E_n^- := \{(z, -\tilde{z}) ; z \in E_n\}$$

Setting $\tilde{E}_n = \{\tilde{v} \mid v \in E_n\}$, one shows exactly as in Lemma 4.4 that

$$E_n \times \tilde{E}_n = E_n^+ \tilde{\oplus} E_n^- .$$

We recall once more that E^+ and E^- are *linear* with respect to the "tilde-sum". Thus, we can define the following "projections"

$$\begin{aligned} P_n^- &: E_n^+ \tilde{\oplus} E_n^- \rightarrow E_n^- , & P_n^-((u, \tilde{v})) &= (\frac{u-v}{2}, -\widetilde{\frac{u-v}{2}}) \\ P_n^+ &: E_n^+ \tilde{\oplus} E_n^- \rightarrow E_n^+ , & P_n^+((u, \tilde{v})) &= (\frac{u+v}{2}, \widetilde{\frac{u+v}{2}}) \end{aligned}$$

which are clearly continuous mappings.

We now restrict the functional I to $E_n \times \tilde{E}_n = E_n^+ \tilde{\oplus} E_n^-$. Consider the set

$$Q_n := \{w \tilde{+} r(e_1, \tilde{e}_1) ; w \in E_n^-, \|w\| \leq R_0 \text{ and } 0 \leq r \leq R_1\} \subset E_n^+ \tilde{\oplus} E_n^- ,$$

where R_0 and R_1 are as in Lemma 5.3. Furthermore, define the class of mappings

$$H_n = \{h \in C(Q_n, E_n^+ \tilde{\oplus} E_n^-) ; h(z) = z \text{ on } \partial Q_n\} ,$$

where ∂Q_n is the boundary of Q_n relative to $E_n^+ \tilde{\oplus} E_n^-$. Finally, set

$$c_n = \inf_{h \in H_n} \max_{z \in Q_n} I(h(z)) .$$

We show

Lemma 5.5 *The sets Q_n and $\partial B_{\rho_0} \cap E_n^+$ link, i.e.*

$$h(Q_n) \cap (\partial B_{\rho_0} \cap E_n^+) \neq 0 , \quad \forall h \in H_n \tag{5.22}$$

Proof. The statement (5.22) is equivalent to saying that

$$\exists (u, \tilde{v}) \in Q_n \text{ such that } \|h(u, \tilde{v})\| = \rho_0 \text{ and } P_n^- h(u, \tilde{v}) = 0 \tag{5.23}$$

Let $(u, \tilde{v}) = w \tilde{+} s(e_1, \tilde{e}_1) \in Q_n$. Define the continuous maps (here we use remark 4.2)

$$\begin{aligned} \psi_t &: Q_n \rightarrow E_n^- \tilde{+} [(e_1, \tilde{e}_1)] \\ \psi_t(w \tilde{+} s(e_1, \tilde{e}_1)) &= \\ &= tP_n^- h((u, \tilde{v})) \tilde{+} (1-t)w \tilde{+} [t\|P_n^+ h((u, \tilde{v}))\| + (1-t)s - \rho_0](e_1, \tilde{e}_1) \end{aligned}$$

Note that for $(u, \tilde{v}) = w \tilde{+} t(e_1, \tilde{e}_1) \in \partial Q_n$, we have

$$\psi_t(w \tilde{+} s(e_1, \tilde{e}_1)) = w \tilde{+} (s - \rho_0)(e_1, \tilde{e}_1) \neq (0, 0), \quad \forall t \in [0, 1],$$

and hence

$$\psi_0(w \tilde{+} s(e_1, \tilde{e}_1)) = w \tilde{+} (s - \rho)(e_1, \tilde{e}_1)$$

is homotopic to

$$\psi_1(w \tilde{+} s(e_1, \tilde{e}_1)) = P_n^- h((u, \tilde{v})) \tilde{+} (\|P_n^+ h((u, \tilde{v}))\| - \rho_0)(e_1, \tilde{e}_1)$$

By the properties of the topological degree on oriented manifolds (see [7]) we have that the degree of the maps ψ_t with respect to Q_n and $(0, 0)$ is well-defined, and that

$$\deg(\psi_1, Q_n, (0, 0)) = \deg(\psi_0, Q_n, (0, 0)) = 1.$$

Hence, there exists an element $(u, \tilde{v}) \in Q_n$ such that $\psi_1(u, \tilde{v}) = (0, 0)$, and hence satisfying (5.23). $\blacksquare$

Choosing ρ_0 as in Lemma 5.2, we now conclude that

$$c_n \geq \sigma_0 > 0, \quad \text{for all } n \in \mathbb{N}.$$

Furthermore, since $id_{E_n^+ \tilde{\oplus} E_n^-} \in \Gamma_n$, we have for $z = r(e_1, \tilde{e}_1) \tilde{+} (u, -\tilde{u}) \in Q_n$

$$c_n \leq \max_{z \in Q_n} I(z) \leq R_1^2 \|e_1\|^2 \leq c R_1^2.$$

Thus, by the Linking Theorem (see [9]), we obtain a PS-sequence, which is bounded in view of Proposition 5.4. Since $E_n^+ \tilde{\oplus} E_n^-$ is finite-dimensional, we therefore get that c_n is a critical level of $I|_{E_n^+ \tilde{\oplus} E_n^-}$, for each $n \in \mathbb{N}$, with a corresponding sequence of critical points $z_n \in E_n^+ \tilde{\oplus} E_n^-$ with $\|z_n\| \leq c$, where c does not depend on n .

5.5 Limit for $n \rightarrow \infty$

By the last subsection we have a sequence $z_n = (u_n, \tilde{v}_n) \in E_n \times E_n$ with

$$I(z_n) = c_n \in [\sigma_0, cR_1^2], \quad \text{and } I'(z_n) = 0. \quad (5.24)$$

By Proposition 5.4 we have $\|z_n\| \leq c$ and hence, for a subsequence, $z_n = (u_n, \tilde{v}_n) \rightarrow z = (u, \tilde{v})$ in $E = W_0^1 L_A \times W_0^1 L_{(\tilde{A})}$. Again by Proposition 5.4 we have $\int_{\Omega} F(u_n) dx \leq c$, $\int_{\Omega} G(\tilde{v}_n) dx \leq c$ and $\int_{\Omega} f(u_n) u_n dx \leq c$, $\int_{\Omega} g(\tilde{v}_n) \tilde{v}_n dx \leq c$. Using Lemma 2.1 in [6] we conclude that

$$f(u_n) \rightarrow f(u) \quad \text{and} \quad g(\tilde{v}_n) \rightarrow g(\tilde{v}) \quad \text{in } L^1(\Omega)$$

Taking arbitrary test functions $(0, \tilde{\eta})$ and $(\zeta, 0)$ in $E_n \times \tilde{E}_n$ we get

$$\int_{\Omega} \nabla u_n \nabla \tilde{\eta} dx = \int_{\Omega} g(\tilde{v}_n) \tilde{\eta} dx, \quad \int_{\Omega} \nabla \tilde{v}_n \nabla \zeta dx = \int_{\Omega} f(u_n) \zeta dx, \quad \forall \eta, \zeta \in E_n. \quad (5.25)$$

Using the fact that $\cup_{n \in \mathbb{N}} (E_n \times \tilde{E}_n)$ is dense in E , we obtain by taking the limit $n \rightarrow \infty$,

$$\int_{\Omega} \nabla u \nabla \tilde{\eta} dx = \int_{\Omega} g(\tilde{v}) \tilde{\eta} dx, \quad \forall \tilde{\eta} \in W^1 L_{\tilde{A}}, \quad \int_{\Omega} \nabla \tilde{v} \nabla \zeta dx = \int_{\Omega} f(u) \zeta dx, \quad \forall \zeta \in W_0^1 L_A.$$

Thus, $(u, \tilde{v}) \in W_0^1 L_A \times W_0^1 L_{(\tilde{A})}$ is a weak solution of problem (1.1).

It remains to show that $(u, \tilde{v})$ is nontrivial; assume by contradiction that $u = 0$, then by the equations (1.1) also $\tilde{v} = 0$. Note that we can find a suitable Δ -regular N-function F_1 with $F_1 \prec\prec \Phi$ and the properties $F(x) \leq F_1(x)$, $f(x) \leq f_1(x)$, $\forall x \in \mathbb{R}^+$. Thus

$$\|u_n\|_{(F_1)} \rightarrow 0, \text{ i.e. } \inf\{\lambda > 0; \int_{\Omega} F_1\left(\frac{u_n}{\lambda}\right) \leq 1\} =: \lambda_n \rightarrow 0$$

Since, for $\lambda_n < 1$ holds $\frac{1}{\lambda_n} \int_{\Omega} F_1(u_n) \leq \int_{\Omega} F_1\left(\frac{u_n}{\lambda_n}\right) \leq 1$, we conclude that

$$\int_{\Omega} F(u_n) \leq \int_{\Omega} F_1(u_n) \leq \lambda_n \rightarrow 0.$$

Since F_1 is Δ -regular, we have $xf_1(x) \leq cF_1(x)$, for some $c > 1$, and hence

$$0 \leq \int_{\Omega} f(u_n)u_n \leq \int_{\Omega} f_1(u_n)u_n \leq c \int_{\Omega} F_1(u_n)dx \rightarrow 0. \quad (5.26)$$

This implies now by (5.25), choosing $\zeta = u_n$, that $\int_{\Omega} \nabla u_n \nabla \tilde{v}_n dx \rightarrow 0$, and thus also $I(u_n, \tilde{v}_n) \rightarrow 0$. But this contradicts that $I(u_n, \tilde{v}_n) \geq \sigma_0 > 0$, for all $n \in \mathbb{N}$.

This concludes the proof of Theorem 1.3.

6 Critical Orlicz pairs near the critical hyperbola

In this section we consider N-functions of the (asymptotic) type

$$\Phi(s) \sim s^{p+1}(\log(1+s))^{\alpha}, \quad (6.27)$$

with $p > 1$ and $\alpha > 0$.

It is natural to expect that the critical Orlicz associate Ψ (i.e. such that (Φ, Ψ) form a critical Orlicz pair) will be given by a N-function Ψ of the asymptotic form

$$\Psi(s) \sim s^{q+1}(\log(1+s))^{-\beta}, \quad (6.28)$$

where q satisfies $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}$, and some relation between α and β . This is indeed so, and the relation between α and β will be given in Proposition 6.3 below.

We begin by showing that functions of type (6.27) and (6.28) satisfy the hypotheses of Theorem 1.3.

Lemma 6.1 *Suppose that $\Phi(t)$ is of the form*

$$\Phi(t) = t^{p+1} g(t),$$

where $p > 1$, and $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying one of the following conditions:

- 1) $g(t)$ increasing
- 2) $g(t) \searrow 0$ and $g(t)t^{\epsilon}$ increasing for large t , for any $\epsilon > 0$

Then Φ is uniformly superquadratic near zero (see Definition 5.1).

Proof. Let $0 < s \leq 1$ and $t > 0$.

1) We have, using that g is increasing,

$$\Phi(st) = (st)^{p+1}g(st) = s^{p+1}t^{p+1}g(t)\frac{g(st)}{g(t)} = s^{p+1}\frac{g(st)}{g(t)}\Phi(t) \leq s^{p+1}\Phi(t)$$

2) We have, for some $0 < \delta < p - 1$

$$\Phi(st) = (st)^{p+1}g(st) = s^{p+1}\frac{g(st)}{g(t)}\Phi(t) = s^{2+\delta}\Phi(t) s^{p-1-\delta}\frac{g(st)}{g(t)} \leq s^{2+\delta}\Phi(t) ;$$

indeed, let $\epsilon = p - 1 - \delta$, and suppose that $t^\epsilon g(t)$ is increasing for $t \geq t_0$. Then we have, since $0 \leq s \leq 1$

$$s^\epsilon \frac{g(st)}{g(t)} \leq \max_{0 \leq t \leq t_0} \frac{g(st)}{g(t)} + \max_{t \geq t_0} \frac{(st)^\epsilon g(st)}{t^\epsilon g(t)} \leq c .$$

■

Lemma 6.2 Suppose that Φ is of class C^1 , and (asymptotically) of the form

$$\Phi(s) \sim c s^{p+1}g(s) , \text{ with } p+1 > \frac{N}{N-2} ,$$

and

$$\lim_{s \rightarrow \infty} \frac{g'(s)}{g(s)} = 0 .$$

Then Φ is θ -regular, with $\theta = p+1$ (see Definition 2.11).

Proof. Indeed, we have

$$\lim_{s \rightarrow \infty} \frac{s \varphi(s)}{\Phi(s)} = \lim_{s \rightarrow \infty} \frac{(p+1)s^{p+1}g(s) + s^{p+1}g'(s)}{s^{p+1}g(s)} = p+1$$

■

Proposition 6.3 Suppose that Φ is (asymptotically) of the form

$$\Phi(s) = c s^{p+1}(\log s)^\alpha , \text{ with } p+1 > \frac{N}{N-2} .$$

Then the associate critical Orlicz function Ψ is (asymptotically) given by

$$\Psi(s) = d s^{q+1}(\log s)^{-\alpha \frac{q+1}{p+1}}$$

with

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N} \tag{6.29}$$

Proof. It is easy to check that (asymptotically)

$$1) \quad \Phi^{-1}(t) \sim c_1 t^{\frac{1}{p+1}} (\log t)^{-\frac{\alpha}{p+1}}$$

$$2) \quad (\Phi^{-1})'(t) \sim c_2 t^{\frac{1}{p+1}-1} (\log t)^{-\frac{\alpha}{p+1}}$$

3) Using definition 2.9:

$$A^{-1}(t) \sim c_3 t^{\frac{N+(p+1)}{N(p+1)}} (\log t)^{-\frac{\alpha}{p+1}}$$

$$4) \quad A(s) \sim c_4 s^{\frac{N(p+1)}{N+(p+1)}} (\log s)^{\alpha \frac{N}{N+(p+1)}}$$

$$5) \quad \tilde{A}(s) \sim c_5 s^{\frac{N(p+1)}{Np-(p+1)}} (\log s)^{-\alpha \frac{N}{Np-(p+1)}}$$

$$6) \quad \tilde{A}^{-1}(t) \sim c_6 t^{\frac{Np-(p+1)}{N(p+1)}} (\log t)^{\frac{\alpha}{p+1}}$$

$$7) \quad \frac{\tilde{A}^{-1}(t)}{t^{1+\frac{1}{N}}} \sim c_6 t^{\frac{-2p-1}{Np}} (\log t)^{\frac{\alpha}{p+1}}$$

8) Using again definition 2.9:

$$\Psi^{-1}(t) \sim c_7 t^{\frac{(N-2)(p+1)-N}{N(p+1)}} (\log s)^{\frac{\alpha}{p+1}}$$

$$9) \quad \Psi(s) \sim c_8 s^{\frac{N(p+1)}{(N-2)(p+1)-N}} (\log s)^{-\frac{\alpha}{p+1} \frac{N(p+1)}{(N-2)(p+1)-N}}$$

Setting $q+1 := \frac{N(p+1)}{(N-2)(p+1)-N}$, once checks that (6.29) holds, and thus finally

$$10) \quad \Psi(s) \sim d s^{q+1} (\log s)^{-\alpha \frac{q+1}{p+1}} \quad \blacksquare$$

We remark that M.A. Krasnoselskiĭ and J.B. Rutickiĭ, in their book on Orlicz spaces [4], Chapter I, § 7, consider the class of N-functions

$$\Phi(s) = cs^{p+1}(\log s)^{\alpha_1}(\log \log s)^{\alpha_2} \dots (\log \log \dots \log s)^{\alpha_k}, \quad \alpha_i \in \mathbb{R}.$$

Repeating the above calculations one shows that the critical Orlicz associates to these functions are given by

$$\Psi(s) = ds^{q+1}(\log s)^{-\beta_1}(\log \log s)^{-\beta_2} \dots (\log \log \dots \log s)^{-\beta_k}.$$

with

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N} \quad \text{and} \quad \beta_i = \alpha_i \frac{q+1}{p+1}, \quad i = 1, \dots, k.$$

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